ਜਗਤ ਗੁਰੂ ਨਾਨਕ ਦੇਵ
ਪੰਜਾਬ ਸਟੇਟ ਓਪਨ ਯੂਨੀਵਰਸਿਟੀ
ਪਟਿਆਲਾ

The Motto of Our University
(SEWA)
SKILL ENHANCEMENT
EMPLOYABILITY
WISDOM
ACCESSIBILITY

JAGAT GURU NANAK DEV
PUNJAB STATE OPEN UNIVERSITY, PATIALA

(Established by Act No. 19 of 2019 of the Legislature of State of Punjab)

B.COM (Hons.)
(Accounting and Taxation)

SEMESTER-II

BCDB31203T
BUSINESS MATHEMATICS AND STATISTICS

Head Quarter: C/28, The Lower Mall, Patiala-147001
Website: www.psou.ac.in

The Study Material has been prepared exclusively under the guidance of Jagat Guru Nanak Dev Punjab State Open University, Patiala, as per the syllabi prepared by Committee of Experts and approved by the Academic Council.

COURSE COORDINATOR AND EDITOR:

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JAGAT GURU NANAK DEV PUNJAB STATE OPEN UNIVERSITY, PATIALA

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PREFACE

Jagat Guru Nanak Dev Punjab State Open University, Patiala was established in December 2019 by Act 19 of the Legislature of State of Punjab. It is the first and only Open University of the State, entrusted with the responsibility of making higher education accessible to all, especially to those sections of society who do not have the means, time or opportunity to pursue regular education.

In keeping with the nature of an Open University, this University provides a flexible education system to suit every need. The time given to complete a programme is double the duration of a regular mode programme. Well-designed study material has been prepared in consultation with experts in their respective fields.

The University offers programmes which have been designed to provide relevant, skillbased and employability-enhancing education. The study material provided in this booklet is self-instructional, with self-assessment exercises, and recommendations for further readings. The syllabus has been divided in sections, and provided as units for simplification.

The University has a network of 10 Learner Support Centres/Study Centres, to enable students to make use of reading facilities, and for curriculum-based counselling and practicals. We, at the University, welcome you to be a part of this instituition of knowledge.

Prof. G.S. Batra
Dean Academic Affairs

B Com (Hons.)
(Accounting and Taxation)
CORE COURSE (CC)

SEMESTER II
BCDB31203T BUSINESS MATHEMATICS AND STATISTICS

MAX. MARKS: 100

EXTERNAL: 70

INTERNAL: 30

PASS: 35%

Objective: **Credits: 6**

The objective of this course is to familiarize students with the applications of mathematics and statistical techniques in business decision-making.

Notes:

1. Use of simple calculator is allowed.
2. Proofs of theorems / formulae are not required.
3. Trigonometric functions are not to be covered.

INSTRUCTIONS FOR THE CANDIDATES:

Candidates are required to attempt any two questions each from the sections A and B of the question paper and any ten short questions from Section C. They have to attempt questions only at one place and only once. Second or subsequent attempts, unless the earlier ones have been crossed out, shall not be evaluated.

Section A

Business Mathematics

Matrices **:** Definition of a matrix. Types of matrices; Algebra of matrices. Calculation of values of determinants up to third order; Adjoint of a matrix; Finding inverse of a matrix through ad joint; Applications of matrices to solution of simple business and economic problems Differential Calculus **:** Mathematical functions and their types – linear, quadratic, polynomial; Concepts of limit and continuity of a function; Concept of differentiation; Rules of differentiation – simple standard forms. Applications of differentiation – elasticity of demand and supply; Maxima and Minima of functions (involving second or third order derivatives) relating to cost, revenue and profit.

Basic Mathematics of Finance **:** Simple and compound interest Rates of interest – nominal, effective and continuous – their interrelationships; Compounding and discounting of a sum using different types of rates

Section B

Business Statistics

Uni-variate Analysis **:** Measures of Central Tendency including arithmetic mean, geometric mean and harmonic mean : properties and applications; mode and median. Partition values - quartiles, deciles, and percentiles. Measures of Variation: absolute and relative. Range, quartile deviation and mean deviation; Variance and Standard deviation: calculation and properties. Bi-variate Analysis **:** Simple Linear Correlation Analysis: Meaning, and measurement. Karl

Pearson's co-efficient and Spearman’s rank correlation. Simple Linear Regression Analysis: Regression equations and estimation. Relationship between correlation and regression coefficients

Index Numbers Analysis **:** Meaning and uses of index numbers; Construction of index numbers: Aggregative and average of relatives – simple and weighted, Tests of adequacy of index numbers, Construction of consumer price indices.

Time series analysis: Components of time series; additive and multiplicative models; Trend analysis: Finding trend by moving average method and Fitting of linear trend line using principle of least squares.

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SEMESTER II

BCDB31203: BUSINESS MATHEMATICS AND STATISTICS

COURSE COORDINATOR AND EDITOR: DR. ROHIT KUMAR

SECTION A

UNIT NO.	UNIT NAME
UNIT 1	MATRICES
UNIT 2	DETERMINANTS
UNIT 3	FUNCTIONS
UNIT 4	LIMIT AND CONTINUITY
UNIT 5	DIFFERENTIATION
UNIT 6	BASIC MATHEMATICS OF FINANCE

SECTION B

UNIT NO.	UNIT NAME
UNIT 7	MEASURES OF CENTRAL TENDENCY
UNIT 8	DISPERSION
UNIT 9	CORRELATION
UNIT 10	REGRESSION
UNIT 11	INDEX NUMBERS
UNIT 12	TIME SERIES ANALYSIS

B. COM (Hons.)

(Accounting and Taxation)
SEMESTER II

COURSE: BUSINESS MATHEMATICS AND STATISTICS

Unit 1 – Matrices

STRUCTURE

1.0 OBJECTIVES

After studying the Unit, students will be able to

- Define the Meaning of a Matrix.
- Understand different types of Matrices.
- Apply Addition, Subtraction and Multiplication of Matrices
- Apply the concept of Matrices to Business and Economic Problems.
- Distinguish between Single and Double Entry System.
- Find out Transpose of a Matrix.
- Understand the meaning of Symmetric, Skew-Symmetric and Orthogonal Matrix.

1.1 INTRODUCTION

Matrices are one of the most important and powerful tools in the mathematics for business. Matrices applications are helpful to solve the linear equations and with the help of this cost estimation, sales projection etc., can be predicted. Basically a matrix consists of a rectangular presentation of numbers arranged systematically in rows and columns describing the various aspects of a phenomenon inter- related in some manner.

For example: Marks obtained by two students, say, Ram and Shyam, in English, Mathematics and Statistics are as follows:

	English	Mathematics	Statistics
Ram	60	80	85
Shyam	65	90	80

These marks may be represented by the following rectangular array enclosed by a pair of brackets [ ],

i.e. $\begin{bmatrix} 60 & 80 & 85 \\ 65 & 90 & 80 \end{bmatrix} \begin{matrix} \leftarrow \text{ First Row} \\ \leftarrow \text{ Second Row} \end{matrix}$

$\quad\uparrow \qquad \uparrow \qquad \uparrow$

First Second Third
Column Column Column

Each horizontal line is called a row and each vertical line is called a column. The first row indicates the marks obtained by Ram in English, Mathematics and Statistics respectively and the second by Shyam in the three respective subjects.

Such a rectangular array is called a Matrix.

1.2 DEFINITION OF A MATRIX

An $m \times n$ matrix is a rectangular array of mn numbers (or elements) arranged in the form of an ordered set of m rows and n columns. A matrix A having m rows and n columns is typically written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

The horizontal lines are called rows and the vertical lines columns.

The numbers $a_{11}, a_{12}, \cdots, a_{mn}$ belonging to the matrix are called is elements.

A matrix having m rows and n columns is said to be of $order\ m \times n$ (read as 'm' by 'n'). The order may be written on right of the matrix, as shown above.

1.3 NOTATIONS

Matrices are denoted by capital letters such as $A, B, C, \ldots, X, Y, Z$and their elements by small letters $a, b, c, \ldots, a_{11}, a_{12}$,etc. There are different nations of enclosing the elements constituting a matrix in common use, $viz.$, [ ], ( ), and { }, but we shall use the first one throughout the chapter. The suffixes of the element a_{ij} depict that the elements lies in ith row and jth column. Note that we always write row number first and column number afterward. Also, note that, for the sake of brevity, the matrix A given above may also be written as$A = [a_{ij}]_{m \times n}$.

1.4 TYPES OF MATRICES

(i) ROW MATRIX: If a matrix has only one row, it is called a *row matrix*. Thus, any $1 \times n$ matrix is called a row matrix, for example,

$$A = [a_1 a_2 a_3 \ldots a_n]$$

is a row matrix of order $1 \times n$.

(ii) COLUMN MATRIX: A matrix consisting of only one column is called a *column matrix*. In other words, any $m \times 1$ matrix is called a column matrix, for example

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

is a column matrix of order $m \times 1$.

(iii) ZERO OR NULL MATRIX (0): If every element of an $m \times n$ matrix is zero, the matrix is called a *zero matrix* of order $m \times n$, and is denoted by $0_{m \times n}$ or 0_{mn} or 0 simply. For example,

$$0_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times3}$$ is a zero matrix of order 2×3.

(iv) SQUARE MATRIX: Any matrix In which the number of rows are equal to the number of columns is called a *Square matrix*. Thus any $n \times n$ matrix is a square matrix of order n.

Generally, we denote the order of a square matrix by a single number n, rather than $n \times n$.

Remark: The elements a_{ij} for which i = j in $A = [a_{ij}]_{n \times n}$ are called the *diadonal elements* and the line along which the elements $a_{11}, a_{12}, \ldots, a_{nn}$ lie is called the *leading diagonal* or *principal diagonal* or *diagonal* simply. In a square matrix the pair of elements a_{ij} and a_{ji} are said to be conjugate elements.

(v) DIAGONAL MATRIX: A square in which all elements except those in the leading diagonal are zero, is called a *diagonal matrix*. Thus, a diagonal matrix of order n will be:

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$0 \quad 0 \quad 0 \quad \cdots \quad a_{nn}$

Sometimes a diagonal matrix of order n with diagonal elements $a_{11}, a_{22}, \ldots, a_{nn}$ is denoted by $A = diag\ (a_{11}, a_{22}, \ldots, a_{nn})$.

(vi) SCALAR MATRIX: A diagonal matrix whose diagonal elements are all equal is called a *scalar matrix*. For example,

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

is a scalar matrix.

(vii) Identity or Unit Matrix (I): A scalar matrix in which each of its diagonal elements is unity is called an *identity* or *unit matrix*.

Thus, a square matrix $A = [a_{ij}]_{n \times n}$ is called identity matrix, if

$$a_{ij} = \begin{cases} 1, & when\ i = j, \\ 0, & when\ i \neq j. \end{cases}$$

An identity matrix of order n is denoted by I_n. Thus, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2.

(viii) TRACE OF A MATRIX: The trace of any square matrix is the sum of its main diagonal elements. For example,

If $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ 2 & 3 & -4 \end{bmatrix}$

Then trace (A) $= 1 + 0 + (-4)$

$= -3$

For a square matrix $A = [a_{ij}]_{n \times n}$

trace of $A = \sum_{i=1}^{m} a_{ii}$

(ix) Triangular Matrix: If every element above or below the leading diagonal is zero, the matrix is called a *triangular matrix*. If the zero element is lie below the leading diagonal, the matrix is called *upper triangular matrix*; If the zero elements is lie above the leading diagonal, the matrix is called *lower triangular matrix*. The matrices A_1 and A_2 given below are the examples of upper and lower triangular matrices respectively:

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$ (upper triangular matrix)

$$A_2 = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$ (lower triangular matrix)

1.5 EQUALITY OF MATRICES

Two matrices are called comparable, if each of them consists of as many rows and columns as the other. Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal, if (i) they have the same order and (iii) have equal corresponding elements throughout ($a_{ij} = b_{ij}$ for every i and j). The equality of matrices A and B.

Thus, the matrices $\begin{bmatrix} 1 & 7 \\ 3 & 5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 9 \end{bmatrix}$ are not comparable while $\begin{bmatrix} 1 & 7 & 8 \\ 3 & 5 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 6 \\ 3 & 2 & 9 \end{bmatrix}$ are comparable but not equal.

The matrices $\begin{bmatrix} 1 & 5 & 9 \\ 3 & 4 & 12 \end{bmatrix}$ and $\begin{bmatrix} 1 & 5 & 3 \times 3 \\ 3 & 2 \times 2 & 3 \times 4 \end{bmatrix}$ are equal.

1.6 SUB MATRIX OF A MATRIX

A matrix which is obtained from a given matrix by deleting any number of rows or columns is called a *sub-matrix* of the given matrix.

For example, the matrix $\begin{bmatrix} 2 & 3 \\ 9 & 3 \end{bmatrix}$ is a sub-matrix of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 3 \end{bmatrix}$. Obtained by deleting 2nd row and 1st column.

1.7 MULTIPLICATION OF A MATRIX BY A SCALAR

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let k be any real number (called scalar). Then the product of k and A denoted by kA is defined to be the $m \times n$ matrix (i, j)th element is ka_{ij}, i.e.,

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}$$

Thus, if $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $3A = \begin{bmatrix} 6 & 9 \\ 12 & 15 \end{bmatrix}$

Thus, we notice that, to get the scalar product each element of the given matrix is multiplied by the given scalar.

PROPERTIES OF SCALAR MULTIPLICATION:

(i) The product of a matrix with a scalar is commutative, i.e., $kA = Ak$.

(ii) If $k = -1$, $(-1)A = [-a_{ij}]$. Generally $(-1)A$ is denoted by $-A$ and is called the negative of matrix A, Thus, $-[a_{ij}] = [-a_{ij}]$.

(iii) If A and B are comparable matrices and k is any scalar, we have $k(A + B) = kA + kB$.

(iv) If k and l are any two scalars and A is any matrix, we have $(k + l)A = kA + lA$. **(v)** If k and l are any two scalars, we have $k(lA) = (kl)A$.

1.8 ADDITION OF MATRICES

The matrices A and B are conformable for addition, if they are comparable. i.e., B has the same number of rows and the same number of column as A. Their sum, denoted by $A + B$, is defined to be the matrix obtained by adding the corresponding elements of A and B.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1. Read the following elements a_{21}, a_{32}, a_{22}, a_{11} in

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 9 & 0 & 4 \\ 8 & 7 & 9 \end{bmatrix}$$

Solution: a_{21} denotes element in second row and first column,

$a_{21} = 9$, $a_{32} = 7$, $a_{22} = 0$, $a_{11} = 4$.

EXAMPLE 2. Construct a 2×3 matrix $A = [a_{ij}]_{2\times 3}$ whose general element is giving by

$$a_{ij} = (i - j)^2/2$$

Solution:

$$a_{11} = \frac{(1-1)^2}{2} = 0,\quad a_{12} = \frac{(1-2)^2}{2} = \frac{1}{2},\quad a_{13} = \frac{(2-1)^2}{2} = \frac{4}{2} = 2$$

$$a_{21} = \frac{(2-1)^2}{2} = \frac{1}{2},\quad a_{22} = \frac{(2-2)^2}{2} = 0,\quad a_{23} = \frac{(2-3)^2}{2} = \frac{1}{2}$$

$$\therefore \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$$

$$\Rightarrow \quad A = \begin{bmatrix} 0 & \frac{1}{2} & 2 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}_{2\times 3}$$

EXAMPLE 3. If $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & -6 & 2 \end{bmatrix}$, find $-3A$.

Solution: $-3A = \begin{bmatrix} -3\times 2 & -3\times 3 & -3\times 5 \\ -3\times 3 & -3\times -6 & -3\times 2 \end{bmatrix} = \begin{bmatrix} -6 & -9 & -15 \\ -9 & 18 & -6 \end{bmatrix}$

For example, if $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 1 & 2 \end{bmatrix}$

and $B = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 3 & 7 \end{bmatrix}$, we have

$$A + B = \begin{bmatrix} 2 + 3 & 3 + 3 & 0 + 5 \\ 3 + 3 & 3 + 1 & 7 + 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 5 \\ 6 & 4 & 9 \end{bmatrix}.$$

In general, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, which is a matrix obtained by adding the elements in the corresponding positions. Thus, from

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$ and

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

We get

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots \\ \cdots & \cdots & \cdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots \end{bmatrix}$$

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[2] .10 PROPERTIES OF MATRIX ADDITION

Suppose A, B, C are three matrices of the same order $m \times n$. Then the matrix addition has following properties:

[3] .1. ASSOCIATIVITy

$$A + (B + C) = (A + B) + C$$

i. *e*., the addition of matrices is associative.

$m \times n$

$a_{1n} + b_{1n}$

$a_{2n} + b_{2n}$]
…

$a_{mn} + b_{mn}$ $m \times n$

1.9 DIFFERENCE OF MATRICES

If A and B are two comparable matrices, then their difference $A - B$ is matrix whose elements are obtained by subtracting the elements of B from the corresponding elements of A.

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$

Then $A - B = [a_{ij} - b_{ij}]_{m \times n}$.

1.2. COMMUTATIVITY

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

$$= [b_{ij} + a_{ij}] = [b_{ij}] + [a_{ij}]$$

$$= B + A$$

i. e., matrix addition is commutative.

1.3. DISTRIBUTIVE LAW

$m(A + B) = mA + mB$ (m being an arbitrary scalar), because

$$m(A + B) = m[a_{ij} + b_{ij}]$$

$$= [m\ a_{ij} + m\ b_{ij}] = m[a_{ij}] +$$

$$m[b_{ij}] = mA + mB.$$

1.4. EXISTENCE OF ADDITIVE IDENTITY

Let A be any $m \times n$ matrix, and 0 the $m \times n$ null matrix. Then, we have

$$A + 0 = 0 + A = A$$

i. e., the null matrix is the identity for the matrix addition.

1.5. Existence of additive inverse

$-A$ is the additive inverse of A, because

$$(-A) + A = [-a_{ij}] + [a_{ij}]$$

$$= [-a_{ij} + a_{ij}] = 0 = A + (-A)$$

Thus, for any matrix A , there exists a unique additive inverse $-A$.

1.6. Cancellation law

$$A + B = A + C \Rightarrow [a_{ij} + b_{ij}] = [a_{ij} + c_{ij}]$$

$$\Rightarrow a_{ij} + b_{ij} = a_{ij} + c_{ij}$$

$$\Rightarrow b_{ij} = c_{ij}$$

$$\Rightarrow [b_{ij}] = [c_{ij}]$$

$$\Rightarrow B = C$$

This is said to be left cancellation.

Similarly, right cancellation, namely,

$B + A = C + A \Rightarrow B = C$ can be proved.

Example 4. If $A = \begin{bmatrix} 1 & 5 & 6 \\ -6 & 7 & 0 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & -5 & 7 \\ 8 & -7 & 7 \end{bmatrix}$**, find** $A + B$ **and** $A - B$**.**

Solution: Here $A + B = \begin{bmatrix} 1 & 5 & 6 \\ -6 & 7 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -5 & 7 \\ 8 & -7 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 1+1 & 5-5 & 6+7 \\ -6+8 & 7-7 & 0+7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 13 \\ 2 & 0 & 7 \end{bmatrix}$$

and $A - B = \begin{bmatrix} 1 & 5 & 6 \\ -6 & 7 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -5 & 7 \\ 8 & -7 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 1-1 & 5-(-5) & 6-7 \\ -6-8 & 7-(-7) & 0-7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 10 & -1 \\ -14 & 14 & -7 \end{bmatrix}$$

Example 5. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$**, evaluate** $3A - 4B$**.**

Solution: $3A - 4B = 3\begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 9 & 3 \\ 0 & -3 & 15 \end{bmatrix} - \begin{bmatrix} 4 & 8 & -24 \\ 0 & -4 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 6-4 & 9-8 & 3-(-24) \\ 0-0 & -3-(-4) & 15-12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 27 \\ 0 & 1 & 3 \end{bmatrix}.$$

Example 6. If X, Y are two matrices given by the eqations

$$X + Y = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \textbf{ and } X - Y = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}, \textit{find } X, Y.$$

Solution: We have

$$X + Y = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \qquad \ldots\text{(i)}$$

$$X - Y = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \qquad \ldots\text{(ii)}$$

By adding equations (i) and (ii),

$$2X = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1+3 & -2+2 \\ 3-1 & 4+0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 4 \end{bmatrix}.$$

$$\therefore \quad X = \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix},$$

From equation (i),

$$Y = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} - X = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & -2-0 \\ 3-1 & 4-2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}.$$

1.11 TEST YOUR UNDERSTANDING (A)

1. Find the elements a_{31}, a_{24}, a_{34}, a_{22} in each of the following matrices given below. Also give their diagonal elements.

$$A = \begin{bmatrix} 8 & 6 & -3 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 9 & 5 & -7 \\ 5 & -3 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 7 & 3 & 5 \\ 2 & 3 & -1 & 0 \\ 3 & 5 & 6 & 8 \\ 4 & 3 & 0 & 0 \\ 2 & 1 & 9 & 8 \end{bmatrix}$$

2. Write the matrix $A = [a_{ij}]$ of order 2×3 whose general elements is given by (i) $a_{ij} = ij$ (ii) $a_{ij} = (-1)^{ij}(i + j)$

3. Find x and y, if

$$\begin{bmatrix} x+y & z \\ 1 & x-y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix}$$

4. If $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$, find the value of $2A + 3B$.

5. Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 5 \end{bmatrix}$, find the matrix C such that $A + 2C = B$.

6. Solve the following equations for A and B:

$$2A - B = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}, \quad 2B + A = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

ANSWERS

1. 3, 0, -7, 3 and 3, 0, 8, 3; the diagonal elements are 8, 3, 8, 0 and 1, 3, 6, 0.

2. (i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 3 & -4 \\ 3 & 4 & 5 \end{bmatrix}$

3. $x = 5$, $y = -2$

4. $\begin{bmatrix} 21 & 22 & 15 \\ 7 & 14 & 23 \end{bmatrix}$

5. $\begin{bmatrix} 1 & -\frac{3}{2} & \frac{5}{2} \\ -\frac{1}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & -2 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$

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1.12 MULTIPLICATION OF MATRICES

The multiplication of one matrix by another is possible, if and only if the number of columns of first matrix is equal to the number of rows of the second matrix. The resulting matrix will have the number of rows equal to those in the first matrix and the number of columns equal to those in the second matrix. Thus, if matrix A is of order $m \times n$ and B is of order $n \times p$, the product AB is possible, i. e., the matrices A and B are conformable for multiplication in the order A, B. The order of the resulting matrix AB will be $m \times p$. The (i, k)th element (i. e., the element laying ith row and kth column) of AB is given by

$$a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}.$$

Thus, to obtain the (i, k)th element of the product AB, we multiply the elements of the ith row of A to the corresponding elements of the kth column of B and add the products thus obtained. The resulting sum is the (i, k)th element of AB.

If AB is denoted by $C = [c_{ij}]$, i. $e.$, $AB = [c_{ij}]$, we have

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{ik} & \cdots & b_{ip} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{j1} & \cdots & b_{jk} & \cdots & b_{jp} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1} & \cdots & b_{nk} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & \cdots & c_{1k} & \cdots & c_{1p} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{i1} & \cdots & c_{ik} & \cdots & c_{ip} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$\begin{bmatrix} c_{m1} & \cdots & c_{mk} & \cdots & c_{mp} \end{bmatrix}$$

Where $c_{ij} = \sum_{j=1}^{n} a_{ij} b_{jk}$. Putting $i = 1, 2, \cdots, m$ and $k = 1, 2, 3, \ldots, p$, all the elements of C will be found.

In the product AB, A is said to be pre-multiplier or pre-factor while B is said to be post-multiplier or post-factor. It is to be noted that in multiplying one matrix by another, unlike ordinary numbers, the placement of matrices as pre factor and post factor is very important. Thus AB is not the same as BA.

1.13 PROPERTIES OF MATRIX MULTIPLICATION

1. MATRIX MULTIPLICATION IS NOT COMMUTATIVE. To verify the above statement, let us take an example. Consider the matrices.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It can easily found that

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ while } BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so that $AB \neq BA$.

This shows that matrix multiplication is not commutative. Actually speaking, for a given pair of matrices A and B, the products AB and BA may not be even comparable. For example, if A is an $m \times n$ matrix and B is an $n \times m$ matrix, AB would be an $m \times m$ matrix and BA would be an $n \times n$ matrix.

It may also happen that for a pair of matrices A and B the product AB may be defined but the product BA may not be defined. For example, it A is an $m \times n$ matrix and B is an $n \times p$ matrix, AB would be an $m \times p$ matrix, but it is not meaningful to talk of BA unless $m = p$.

Note 1. It is worthwhile to note that the statement ‘matrix multiplication is not commutative’ does not mean that there are no matrices A and B such that $AB = BA$. It simply means that generally $AB \neq BA$. Thus, we wish to convey that there do exist some pairs of matrices A and B for which $AB = BA$.

Note 2. It is also to be noted that in matrices, $AB = 0$ need not always imply that either $A = 0$ or $B = 0$, This will be clear, if we consider the matrices,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

For these matrices, $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but none of A and B is zero matrix.

Note 3. The familiar cancellation law of multiplication for numbers fails to be true for matrix multiplication. Below we give the properties which hold good for matrices.

2. ASSOCIATIVE LAW: Let A, B and C be the matrices of suitable size for the products $A(BC)$ and $(AB)C$ to exist. Then, $A(BC) = (AB)C$.

3. DISTRIBUTIVE LAW: $A(B + C) = AB + AC$, (left distributive) and $(B + C)D = BD + CD$ (right distributive), provided that the matrices A, B, C and are of the sizes that they are conformable for the operations involved so that the above relations are meaningful.

4. MULTIPLICATION OF A MATRIX BY A UNIT MATRIX: If A is square matrix of order $n \times n$ and I is the unit matrix of the same order, we get

$$I = A = IA.$$

5. MULTIPLICATION OF A MATRIX BY ITSELF: The product A, A is defined, if the number of column is equal to the number of rows of A, i. $e.$, if A is a square matrix and in that case $A.A = A^2$ will also be a square matrix of the same type. Also,

$$A.\ A.\ A = A^2A = A^3.$$

Similarly, $A. A. A \ldots . n$ times$= A^n$.

Note: If I is a unit matrix, we have $I = I^2 = I^3 = \cdots = I^n$.

Example 7. If $A = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$,

Obtain the product AB and BA and show than $AB \neq BA$.

Solution:

$$AB = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}_{3\times3} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}_{3\times3}$$

Number of columns of A = Number of rows of B

$\therefore$ AB is defined

$$AB = \begin{bmatrix} 0+0+3 & 0+2+6 & 4+4+0 \\ 0+0-1 & 0+3-2 & 4+6+0 \\ 0+0+2 & 0+1+4 & -6+2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 & 8 \\ -1 & 1 & 10 \\ 2 & 5 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}_{3\times3} \begin{bmatrix} 2 & 2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}_{3\times3}$$

$$= \begin{bmatrix} 0+0-6 & 0+0+2 & 0+0+4 \\ 0+2-6 & 0+3+2 & 0-1+4 \\ 2+4+0 & 2+6+0 & 3-2+0 \end{bmatrix}_{3\times3}$$

$$= \begin{bmatrix} -6 & 2 & 4 \\ -4 & 5 & 3 \\ 6 & 8 & 1 \end{bmatrix}$$

Hence $AB \neq BA$.

Example 8. Write down the products AB and BA of the two matrices A and B, where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solution: Since A is a 1×4 matrix and B is a 4×1 matrix. ABwill be a 1×1 matrix.

$$AB = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= [1.1 + 2.2 + 3.3 + 4.4] = [30]$$

BA will be a 4×4 matrix.

$$BA = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 1 & 2 \times 1 & 3 \times 1 & 4 \times 1 \\ 1 \times 2 & 2 \times 2 & 3 \times 2 & 4 \times 2 \\ 1 \times 3 & 2 \times 3 & 3 \times 3 & 4 \times 3 \\ 1 \times 4 & 2 \times 4 & 3 \times 4 & 4 \times 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

Example 9. *Evaluate* $A^2 - 4A - 5I$, *where*

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: We have

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8+0 & 8-8+0 \\ 8-8+0 & 9-4-5 & 8-8+0 \\ 8-8+0 & 8-8+0 & 9-4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Where $O$is the null matrix.

Example 10. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **and** $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$**, show that**

$$A^2 - (a+d)A = (bc - ad)I.$$

Solution: $A^2 = A.A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$$

$$A^2 - (a+d)A = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} - \begin{bmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{bmatrix}$$

$$= \begin{bmatrix} bc - ad & 0 \\ 0 & bc - ad \end{bmatrix} = (bc - ad)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= (bc - ad)I.$$

Example 11. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$**,** $B = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$ **and** $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$**, show that** $A(B + C) = AB + AC$**.**

Solution: We have

$$B + C = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 0-1 \\ 2+0 & -3+1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}.$$

$$A(B + C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4 & -1-4 \\ 6+8 & -3-8 \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ 14 & -11 \end{bmatrix}, \quad \ldots\text{(i)}$$

Again,

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4 & 0-6 \\ 3+8 & 0-12 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 11 & -12 \end{bmatrix}$$

and

$$AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & -1+2 \\ 3+0 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix},$$

$$\therefore \quad AB + AC = \begin{bmatrix} 5 & -6 \\ 11 & -12 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ 14 & -11 \end{bmatrix} \quad \ldots\text{(ii)}$$

From (i) and (ii), we have

$$A(B + C) = AB + AC.$$

Example 12. If $[1 \quad 1 \quad x]\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = 0$ **find** x

Solution: $[1 \quad 1 \quad x]\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

$= [1 \times 1 + 1 \times 0 + x \times 2 \quad 1 \times 0 + 1 \times 2 + x \times 1 \quad 1 \times 2 + 1 \times 1 + x \times 0]$

$= [1 + 0 + 2x \quad 0 + 2 + x \quad 2 + 1 + 0]$

$= [2x + 1 \quad 2 + x \quad 3]$

Now, $[2x + 1 \quad 2 + x \quad 3] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$

$\Rightarrow \quad 2x + 1 + 2 + x + 3 = 0$

$\Rightarrow \quad 3x + 6 = 0$ or

$x = \frac{-6}{3} = -2$

13. $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$**,** $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ **and** $(A + B)^2 = A^2 + B^2$

Example find a **and** b**.**

Solution: We have

$$A^2 = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1-2 & -1+1 \\ 2-2 & -2+1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and $$B^2 = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} \times \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} a^2 + b & a - 1 \\ ab - b & b + 1 \end{bmatrix}$$

$$A^2 + B^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} a^2 + b & a - 1 \\ ab - b & b + 1 \end{bmatrix} = \begin{bmatrix} a^2 + b - 1 & a - 1 \\ ab - b & b \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} 1 + a & -1 + 1 \\ 2 + b & -1 - 1 \end{bmatrix} = \begin{bmatrix} 1 + a & 0 \\ 2 + b & -2 \end{bmatrix}$$

$$+ B)^2 = \begin{bmatrix} 1 + a & 0 \\ 2 + b & -2 \end{bmatrix} \times \begin{bmatrix} 1 + a & 0 \\ 2 + b & -2 \end{bmatrix} = \begin{bmatrix} (1 + a)^2 + 0 & 0 + 0 \\ (2 + b)(1 + a) - 2(2 + b) & 0 + 4 \end{bmatrix}$$

$\therefore$ …(1)

Also

$\therefore \quad (A$

$$= \begin{bmatrix} (1 + a)^2 & 0 \\ (2 + b)(a - 1) & 4 \end{bmatrix}. \qquad \ldots(2)$$

Now given that $(A+B)^2 = A^2 + B^2$.

Hence from (1) and (2), we get

$$\begin{bmatrix} (1+a)^2 & 0 \\ (2+b)(a-1) & 4 \end{bmatrix} = \begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix}$$

or $a - 1 = 0$ and $b = 4$.

Hence $a = 1, b = 4$.

1.14 APPLICATIONS OF MATRICES TO BUSINESS AND ECONOMIC PROBLEMS

Example 14. A manufacturer produces three items P, O and R and sells them in two markets I and II. Annual sales are given below:

	P	O	R
I	***6,000***	***2,000***	***3,000***
II	***8,000***	***4,000***	***2,000***

If sales price of each unit of P, Q, R is Rs. 4, Rs. 3, Rs. 2 respectively, then find the total revenue of each market using matrix.

Solution: Let $A = \begin{matrix} I \\ II \end{matrix}\begin{bmatrix} 6{,}000 & 2{,}000 & 3{,}000 \\ 8{,}000 & 4{,}000 & 2{,}000 \end{bmatrix}$ is the sales matrix and

$B = \begin{matrix} P \\ Q \\ R \end{matrix}\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ is the price matrix

$\therefore$ Revenue matrix $AB = \begin{bmatrix} 6{,}000 & 2{,}000 & 3{,}000 \\ 8{,}000 & 4{,}000 & 2{,}000 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 6{,}000 \times 4 + 2{,}000 \times 3 + 3{,}000 \times 2 \\ 8{,}000 \times 4 + 4{,}000 \times 3 + 2{,}000 \times 2 \end{bmatrix} = \begin{bmatrix} 36{,}000 \\ 48{,}000 \end{bmatrix}$$

Hence

Total Revenue of Market I = Rs. 36,000

Total Revenue of Market II = Rs. 48,000

1.15 TEST YOUR UNDERSTANDING (B)

1. If $A = \begin{bmatrix} 3 & 6 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 7 \\ 2 \end{bmatrix}$, find AB and BA.

2. If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix}$, find AB and BA. Is $= BA$?

3. If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, find AB and show that $AB = BA$.

4. When $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$ and $B = \begin{bmatrix} i & -1 \\ -1 & -i \end{bmatrix}$ and i = √(−1), determine AB. Also compute BA.

5. If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -1 \\ -3 & 3 & 3 \end{bmatrix}$,

Show that AB and CA are the null matrices but BA and AC are not the null matrices.

6. If $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ 3 & 5 \end{bmatrix}$,

Find a and b such that $AB = BA$. Also compute $3A + 5B$.

7. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$, show that $A(B + C) = AB + AC$.

8. If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, prove by mathematical induction that $A^n = \begin{bmatrix} 1 & na \\ 0 & 1 \end{bmatrix}$

B

9. Given $A =$ [1 2 10 −13] , = [10 32] and $C =$ [12 20 −23 −41] ,

3 −1 2 −1 4

Show that $(AB)C = A(BC)$.

10. If $A = \begin{bmatrix} -5 & -2 & 1 \\ 4 & 3 & 3 \\ -6 & 6 & -2 \end{bmatrix}$

Find the matrix B such that $A + B =$ unit matrix.

11. There are two families A and B. In family A, there are 4 men, 6 women and 2 children; and in family B there are 2 men, 2 women and 4 children. The recommended daily requirement for calories is:

Calories: Man 2,400; Woman 1,900; Child 1,800

Protein: Man 55 gm; Woman 45 gm; Child 33 gm

Calculate the total requirements of calories and proteins for each of the two families using matrix method.

12. The co-operative store of a particular school has 10 dozen books of physics, 8 dozen of chemistry books and 5 dozen of mathematics books. Their selling price are Rs. 65.70, Rs. 43.20 and Rs. 76.50 respectively. Find by matrix method the total amount, the store will receive from selling all three items.

Answers

1. $AB = [44]$, $BA = \begin{bmatrix} 12 & 24 & -20 \\ 21 & 42 & -35 \end{bmatrix}$, **2.** $AB = \begin{bmatrix} -13 & 8 \\ -8 & 5 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 2 \\ 4 & -9 \end{bmatrix}$, −

3. $AB = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$ $\begin{bmatrix} 6 & 12 & -10 \\ 0 & 0 & -2 \end{bmatrix}$ **4.** $AB = \begin{bmatrix} 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 2i & -2 \\ -2i & \end{bmatrix}$,

6. $a = 65, b = 15$, $\begin{bmatrix} 41.5 & 13.5 \\ 27 & 28 \end{bmatrix}$

10. $\begin{bmatrix} 6 & 2 & -1 \\ -4 & -2 & -3 \\ 6 & -6 & -1 \end{bmatrix}$

11. Calories for family A and B are 24,600 and 15,800 and proteins are 556 gms and 332 gms respectively.

12. Rs. 16,621.20

1.16 TRANSPOSE OF A MATRIX

A matrix obtained by interchanging the corresponding rows and columns of a given matrix A is called the **transpose matrix** of A**.** The transpose of a matrix is denoted by A^T or A'.

For example,

If $A = [1\ 5]$, then $A' = [^1]\ 5$

If $A = [\ ^2\ ]$, then $A' = [2 \quad -3]$
-3

If $A = [^{1\ 2}]$, then 1

3 A'

$= [\quad]$

3 4 2 4

Properties of the transpose matrix

(i) $(A')' = A$

(ii) $(A + B)' = A' + B'$

(iii) $(\lambda A)' = \lambda A'$

(iv) If A and B are two matrices which are conformable for multiplication, then $(AB)' = B'A'$ This

is called '$Reversal\ Law$'.

Example 15.

1 2 −3

If $A = [3 \quad 0 \quad 2\ ]$

4 5 0

1 0 0

and $B = [2\ 1 \quad 0]$

0 1 3

Then, verify that $(AB)^T = B^T A^T$

Solution.

1 2

$A = [3\ 0$

$4 \quad 5 \quad 0$

$1 \quad 2 \quad -3$

$AB = [3 \quad 0 \quad 2\ ]$

$4 \quad 5 \quad 0$

$(1 + 4 + 0)$

$= [\ (3 + 0 + 0)$

$(4 + 10 +$

From (i) and (ii),

$5 \quad -1$

$= [\ 3 \quad 2$

$14 \quad 5$

$-3 \qquad 1 \quad 0 \quad 0$

$2\ ] \quad$ and $B = [2 \quad 1 \quad 0]$

$_{3\times3} \quad 0 \quad 1 \quad 3\ _{3\times3}$

$5 \quad 3 \quad 14$

$(AB)^T = [-1 \quad 2 \quad 5$

$-9 \quad 6 \quad 0$

$1 \quad 0 \quad 0$

$[2 \quad 1 \quad 0]$

$0 \quad 1 \quad 3$

$1 \quad 3 \quad 4$

$A^T = [\ 2 \quad 0 \quad 5]$

$-3 \quad 2 \quad 0$

$(0 + 2 - 3) \quad (0 + 0 - 9)$

$(0 + 0 + 2) \quad (0 + 0 + 6)]$

$0) \quad (0 + 5 + 0) \quad (0 + 0 + 0)$

$1 \quad 2 \quad 0$

$B^T A^T = [0 \quad 1 \quad 1]\ [$

$0 \quad 0 \quad 3$

$(1 + 4 - 0)$

$= [(0 + 2 - 3)$

$(0 + 0 - 9)$

-9

$6\ ]$

0

$5 \quad 3 \quad 14$

$= [-1 \quad 2 \quad 5$

$-9 \quad 6 \quad 0$

$]$ …(i)

$1 \quad 2 \quad 0$

$, B^T = [0 \quad 1 \quad 1]$

$_{3\times3} \quad 0 \quad 0 \quad 3\ _{3\times3}$

$1 \quad 3 \quad 4$

$2 \quad 0 \quad 5]$

$-3 \quad 2 \quad 0$

$(3 + 0 + 0) \quad (4 + 10 + 0)$

$(0 + 0 + 2)\ (0 + 5 + 0)\ ]\ (0 + 0$

$+ 6)\ (0 + 0 + 0)$

$]$ …(ii)

$$(AB)^T = B^T A^T$$

1.17 SYMMETRIC MATRIX

A square matrix A such that $A' = A$ is called symmetric matrix, i. e., matrix $[a_{ij}]$ is symmetric provided $a_{ij} = a_{ji}$ for all values of i and j.

For example:

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$ is a symmetric matrix.

1.18 SKEW-SYMMETRIC MATRIX

A square matrix A such that $A' = -A$ is called skew-symmetric matrix, i. e., matrix $[a_{ij}]$ is skew-symmetric provided $a_{ij} = -a_{ji}$ for all values ofi and j.

For example:

$$A = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$$ is a skew-symmetric matrix.

Remarks. In a skew-symmetric matrix, we have $a_{ij} = -a_{ji}$. For diagonal elements

$a_{ij} = -a_{ji}$, i. $e.$, $2a_{ij} = 0$, or $a_{ij} = 0$.

Thus every diagonal element of a skew-symmetric matrix is zero.

Example 16.

Show that every matrix can be unequally expressed as the sum of a symmetric and a skew-symmetric matrix.

Solution: Let A be any square matrix.

Now, we have

$$A = \frac{1}{2}A + \frac{1}{2}A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') \ldots \text{(i)}$$

$$(A + A')' = A' + (A')' = A' + A = A + A'$$

Now,

(since matrix addition is commutative)

and $(A - A')' = A' - (A')' = A' - A = -(A - A')$ Hence $(A + A')$ is symmetric and $(A - A')$ is skew-symmetric.

Consequently $\frac{1}{2}(A + A') = P$ (say) is a symmetric matrix and $\frac{1}{2}(A - A') = Q$ (say) is a skew-symmetric matrix.

Hence $\quad A = P + Q.$

Thus, any square matrix, can be expressed as the sum of symmetric and skew-symmetric matrix. **Uniqueness.** To show that this representation is unique, let us suppose that another representation $A = R + S$ is possible, where R is symmetric and S is skew-symmetric, i. *e*. $R = R'$ and $= S'$.

Now $\quad A' = (R + S)' = R' + S' = R - S$

Also, $\quad A + A' = (R + S) + (R - S) = 2R$

and $\quad A - A' = (R + S) - (R - S) = 2S$

or $\quad R = \frac{1}{2}(A + A')$ and $S = \frac{1}{2}(A - A')$

Hence $\quad A = \frac{1}{2}(A - A') + \frac{1}{2}(A - A')$, is a unique representation.

Example 17. Express $\begin{bmatrix} 2 & 6 & -8 \\ 4 & 2 & 1 \\ -8 & 6 & 13 \end{bmatrix}$ **as a sum of a symmetric and skew-symmetric matrix.**

Solution: Let

$$A = \begin{bmatrix} 2 & 6 & -8 \\ 4 & 2 & 1 \\ -8 & 6 & 13 \end{bmatrix}$$

$$\therefore \quad A' = \begin{bmatrix} 2 & 4 & -8 \\ 6 & 2 & 6 \\ -8 & 1 & 13 \end{bmatrix}$$

$$A + A' = \begin{bmatrix} 4 & 10 & -16 \\ 10 & 4 & 7 \\ -16 & 7 & 26 \end{bmatrix}$$

$$\frac{1}{2}(A + A') = \frac{1}{2}\begin{bmatrix} 4 & 10 & -16 \\ 10 & 4 & 7 \\ -16 & 7 & 26 \end{bmatrix}$$

∴

∴

Which is a symmetric matrix.

$$\frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -5 \\ 0 & 5 & 0 \end{bmatrix}$$

Again

Which is a skew-symmetric matrix. Now

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

$$A = \frac{1}{2}\begin{bmatrix} 4 & 10 & -16 \\ 10 & 4 & 7 \\ -16 & 7 & 26 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -5 \\ 0 & 5 & 0 \end{bmatrix}$$

i. *e*.,

Example 18. ***If A and B are both symmetric then, show that AB is symmetric iff A*** **and** ***B commute.***

Solution: Since A and B are symmetric, we have

$$A' = A \text{ and } B' = B$$

Then $(AB)' = B'A'$ (reversal law)

$= BA = AB$, iff A and B commute Thus

AB is symmetric iff A and B commute.

1.19 ORTHOGONAL MATRIX

A square matrix A is said to be orthogonal if $A'A = AA' = I$

Now, we know that

$$|A'| = |A|$$

Also $|A'A| = |A'||A|$ or $|I| = |A|^2$ or $|A|^2 = 1$

This shows that the matrix A should be non_singular and invertible, if it is orthogonal matrix. Hence the condition

$A'A = I$ Implies that $A^{-1} = A'$ **Example**

19. ***Verify that***

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

is orthogonal. **Solution:**

We have

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

$$A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

Then

We have

$$AA' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1+4+4 & 2+2-4 & -2+4-2 \\ 2+2-4 & 4+1+4 & -4+2+2 \\ -2+4-2 & -4+2+2 & 4+4+1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

Hence, by definition, matrix A is orthogonal.

1.20 TEST YOUR UNDERSTANDING(C)

1. If A and B are symmetric, then show that $A + B$ is symmetric.

2. If $A = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$, verify that $(6A)' = 6A'$.

3. Given : $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$ obtain the matrix $2A' + 3B'$.

4. Express $\begin{bmatrix} 2 & 1 & -11 \\ 6 & 13 & 7 \\ -4 & -2 & 1 \end{bmatrix}$ as a sum of a symmetric and skew-symmetric matrix.

Answers

3. $\begin{bmatrix} 10 & 5 \\ -3 & 10 \end{bmatrix}$

4. $\frac{1}{2}\begin{bmatrix} 4 & 7 & -15 \\ 7 & 26 & 5 \\ -15 & 5 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -5 & -7 \\ 5 & 0 & 9 \\ 7 & -9 & 0 \end{bmatrix}$

1.21 LET US SUM UP

- A Matrix is a particular type of arrangement of m n numbers which are arranged in the form of an ordered set of m rows and n columns.
- There are different types of Matrices like Row, Column, Square, Rectangular, Null, Diagonal, Scalar, Unit, Triangular matrix etc.
- Matrix algebra includes addition of matrices, difference of matrices, multiplication of a matrix by a scalar and matrix multiplication.
- We can obtain the Transpose of a Matrix by interchanging the corresponding rows and columns of a given matrix.

1.22 KEY TERMS

- **ROW MATRIX:** If a matrix has only one row, it is called a $row\ matrix$.
- **COLUMN MATRIX:** A matrix consisting of only one column is called a $column\ matrix$.
- **ZERO OR NULL MATRIX (0):** If every element of an $m \times n$ matrix is zero, the matrix is called a $zero\ matrix$.
- **SQUARE MATRIX:** Any matrix In which the number of rows are equal to the number of columns is called a $Square\ matrix$.
- **DIAGONAL MATRIX:** A square in which all elements except those in the leading diagonal are zero, is called a $diagonal\ matrix$.

- **SCALAR MATRIX:** A diagonal matrix whose diagonal elements are all equal is called a *scalar matrix*.
- **IDENTITY OR UNIT MATRIX (I):** A scalar matrix in which each of its diagonal elements is unity is called an *identity* or *unit matrix*.
- **TRACE OF A MATRIX:** The trace of any square matrix is the sum of its main diagonal elements.
- **TRIANGULAR MATRIX:** If every element above or below the leading diagonal is zero, the matrix is called a *triangular matrix*.
- **TRANSPOSE OF A MATRIX**: A matrix obtained by interchanging the corresponding rows and columns of a given matrix.
- **SYMMETRIC MATRIX:** A matrix which is equal to its own transpose.
- **SKEW SYMMETRIC MATRIX:** A matrix which is equal to negative of its own transpose.

1.23 FURTHER READINGS

1. Mizrahi and John Sullivan. Mathematics for Business and Social Sciences. Wiley and Sons.
2. N. D. Vohra, Business Mathematics and Statistics, McGraw Hill Education (India) Pvt Ltd
3. J.K. Thukral, Mathematics for Business Studies, Mayur Publications
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B. COM (Hons.)

(Accounting and Taxation)

SEMESTER II

COURSE: BUSINESS MATHEMATICS AND STATISTICS

Unit 2 – Determinants

STRUCTURE

2.0 OBJECTIVES

After studying the Unit, students will be able to

- Define the Meaning Determinant.
- Calculate the Determinant of a Square Matrix.
- Implement the properties of Determinants.
- Understand the difference between Minors and Co-Factors.
- Find the Inverse of a Matrix.
- Solve the Linear Equations with help of Matrices.
- Solve the Linear Equations with help of Determinants

2.1 DETERMINANT

To every square matrix $A = [a_{ij}]_{m \times m}$, we associate a number called determinant of the matrix A. It is denoted by determinant A or $|A|$. The matrix which is not square does not possess determinant.

FINDING VALUE OF A DETERMINANT

1) **Determinant of order 1,** i. e. of 1×1 matrix

   The value of the determinant of order one is the number of which the determinant is formed.

   Thus, $|a_{11}| = a_{11}$

   For example $|-2| = -2$

2) **Determinant of order 2,** i.e. of 2×2 matrix

   The value of the determinant of order two is found as under :

$a \quad a$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{12}a_{21})$$

$$\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix}$$

$$= (2)(-1) - 3 \times 0$$

$$= -2 - 0 = -2$$

For example

3) **Determinant of order 3,** i.e. of 3×3 matrix

The value of the determinant of order three is found by expanding the determinant by any row or any column.

For example $|A| = \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Expanding it by 1st row, we get

$$\Delta = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Similarly we can expand it by any other row or any column, taking care of the fact the signs meddler for adding the terms will follow the following pattern

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Thus, by expanding the determinant by, say, second column, we have

$$\Delta = -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$= -a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) - a_{32}(a_{11}a_{23} - a_{13}a_{21})$$

Thus if we have

$$A = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 5 & 2 \\ 4 & -3 & 1 \end{vmatrix}$$

then

$$|A| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 5 & 2 \\ 4 & -3 & 1 \end{vmatrix}$$

Expanding it by 'R_1', we get

$$2\begin{vmatrix} 5 & 2 \\ -3 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} -1 & 5 \\ 4 & -3 \end{vmatrix}$$

$$= 2(5+6) + 1(-1-8) + 0(3-20)$$

$$= 2(11) + 1(-9) + 0(-17)$$

$$= 22 - 9 + 0 = 13$$

Example 1. Find the value of determinant

$$\begin{vmatrix} 3 & -5 & 8 \\ 6 & -4 & -3 \\ 4 & 2 & 0 \end{vmatrix} \; by\ expanding\ \text{it}\ by$$

(i) $second\ row\ and$ (ii) $third\ column$.

Solution: (i) Expanding by the second row,

$$\begin{vmatrix} 3 & -5 & 8 \\ 6 & -4 & -3 \\ 4 & 2 & 0 \end{vmatrix} = -6\begin{vmatrix} -5 & 8 \\ 2 & 0 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 8 \\ 4 & 0 \end{vmatrix} - (-3)\begin{vmatrix} 3 & -5 \\ 4 & 2 \end{vmatrix}$$

$$= -6(-16) - 4(-32) + 3(26) =$$
$$96 + 128 + 78 = 302$$

(ii) Expanding by the third column,

$$\begin{vmatrix} 3 & -5 & 8 \\ 6 & -4 & -3 \\ 4 & 2 & 0 \end{vmatrix} = 8\begin{vmatrix} 6 & -4 \\ 4 & 2 \end{vmatrix} - (-3)\begin{vmatrix} 3 & -5 \\ 4 & 2 \end{vmatrix} + 0\begin{vmatrix} 3 & -5 \\ 6 & -4 \end{vmatrix}$$

$$= 8(28) - 3(26) + 0$$

$= 224 + 78 = 302$, same as before.

2.2 PROPERTIES OF DETERMINANTS

The following are the important properties of determinants. The students are advised to verify these properties on their own.

(i) If any two rows (or columns) of a determinant are interchanged, the sign of the determinant is changed, the absolute value remaining unaltered, For example,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & a_3 & a_2 \\ b_1 & b_3 & b_2 \\ c_1 & c_3 & c_2 \end{vmatrix}$$

In this example the second and third columns have been interchanged.

(ii) If every element in any row (or column) of a determinant is multiplied by the same scalar c, the determinant thus obtained is c times the original determinant. Thus,

$$\begin{vmatrix} a_1 & ca_2 & a_3 \\ b_1 & cb_2 & b_3 \\ c_1 & cc_2 & c_3 \end{vmatrix} = c \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

(iii) If a determinant rows are changed into columns of columns into rows, the value of the determinant remains unchanged. Thus,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = c \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(iv) If any two rows (or columns) of a determinant are identical, the value of the determinant is zero. Thus,

$$\begin{vmatrix} a_1 & b_2 & c_3 \\ a_1 & b_1 & b_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0, \begin{vmatrix} 3 & 9 & 5 \\ 2 & 4 & 7 \\ 1 & 3 & 6 \end{vmatrix} = 3 \begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & 7 \\ 1 & 1 & 6 \end{vmatrix} = 3 \times 0 = 0.$$

(v) If each element of any row or any column is the sum (or difference) of two quantities, the determinant can be expressed as the sum (or difference) of two determinants of the same order, as given below :

$$\begin{vmatrix} a_1 & a_2 + \alpha & a_3 \\ b_1 & b_2 + \beta & b_3 \\ c_1 & c_2 + \gamma & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \alpha & a_3 \\ b_1 & \beta & b_3 \\ c_1 & \gamma & c_3 \end{vmatrix}.$$

(vi) If any row (or column) or a multiple thereof is added to or subtracted from any other row (or column), the value of the determinant remain unchanged. Thus,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 + ka_1 \\ b_1 & b_2 & b_3 + kb_2 \\ c_1 & c_2 & c_3 + kc_2 \end{vmatrix}$$

Example 2. Without expanding prove that

$$\begin{vmatrix} 6 & 2 & 7 \\ 15 & 5 & 5 \\ 21 & 7 & 3 \end{vmatrix} = 0$$

Solution: Let

$$\Delta = \begin{vmatrix} 6 & 2 & 7 \\ 15 & 5 & 5 \\ 21 & 7 & 3 \end{vmatrix}$$

Taking ‘3’ common from C_1, we have

$$\Delta = 3\begin{vmatrix} 2 & 2 & 7 \\ 5 & 5 & 5 \\ 7 & 7 & 3 \end{vmatrix} = 3(0) = 0 \quad (\because C_1 \text{ and } C_2 \text{ are identical})$$

Example 3. Withoutexpanding evaluate the determinant

$$\begin{vmatrix} 42 & 2 & 5 \\ 80 & 8 & 9 \\ 30 & 6 & 3 \end{vmatrix}$$

Solution: Let

$$\Delta = \begin{vmatrix} 42 & 2 & 5 \\ 80 & 8 & 9 \\ 30 & 6 & 3 \end{vmatrix}$$

Applying $C_1 = C_1 + (-8)C_3$, we get

$$\Delta = \begin{vmatrix} 2 & 2 & 5 \\ 8 & 8 & 9 \\ 6 & 6 & 3 \end{vmatrix}$$

$\Rightarrow \quad \Delta = 0 \qquad (\because C_1 \text{ and } C_2 \text{ are identical})$

Example 4. Prove that

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \end{vmatrix} = 0$$

$b + c \quad a + c \quad a + b$

Solution:

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b + c & a + c & a + b \end{vmatrix}$$

Applying $R_1 = R_1 + R_2$, we get

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ a + b + c & b + a + c & c + a + b \\ b + c & a + c & a + b \end{vmatrix}$$

$$= (a + b + c) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ b + c & a + c & a + b \end{vmatrix} \text{ \{taking } (a + b + c) \text{ common from } R_2$$

$$= (a + b + c) \times (0) \qquad (\because R_1 \text{ and } R_2 \text{ are identical})$$

$$= 0$$

Hence proved.

Example 5. Prove that

$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Solution:

$$L.H.S. = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \frac{1}{abc} \begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix}$$

(by multiplying R_1, R_2, R_3 by abc respectively)

$$= \frac{abc}{abc} \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \quad \text{(taking } abc \text{ common from } C_2 \text{)}$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = R.H.S.$$

Example 6. Without expansion, prove that

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0$$

Solution: Let

$$\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix}$$

$$= (-1)^3 \begin{vmatrix} 0 & -a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix}$$ (by taking (-1) common from each of 3 rows) (Interchanging rows and columns)

$$= - \begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix}$$

$$= -\Delta$$

$$\Delta = -\Delta$$

$\Rightarrow$ $2\Delta = 0$

$\Rightarrow$ $\Delta = 0$

$\Rightarrow$

i. *e*. $\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0$

Example 7. Prove that

1 1 1

$| a \quad b \quad c | = (a-b)(b-c)(c-a).$ 1 1 1 0

$a^2 \quad b^2 \quad c^2$ 0 1

Solution:

C 1

$| a \quad b \quad c | = | a-b \quad b-c \quad c |$ where $_1 = C_1 - C_2$ =

$a^2 \quad b^2 \quad c^2 \quad a^2 - b^2 \quad b^2 - c^2$ $C\ _2 = C_2 - C_3$

c^2

(

$a-b)(b-c) \mid 1\ 1\ c$

$| a+b\ b+c\ c^2$

1 1 $ac \quad bc \quad -c^2$

$= (a-b)(b-c)\ \Big| \quad \Big|$ **Solution:**

$a+b \quad b+$

$c = (a-b)(b-c)(b+c-a-$ $-a \quad b \quad c$

$b)$ L.H.S.$= abc \mid a$

$-b \quad c \mid$

$a \quad b \quad -c$

$= (a-b)(b-c)(c-a).$

on taking out a, b and c common

Example 8. Prove that (fromR_1, R_2andR_3repectively)

$-a^2 \quad ab \quad ac$

$| ba \quad -b^2 \quad bc | = 4a^2b^2c^2,$

$-1 \quad 1 \quad 1$ on taking out a, b and c common

$= a^2b^2c^2 \mid \quad \mid \quad (\quad)$

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$ from C_1, C_2 and C_3 repectively

$$= a^2b^2c^2 \begin{vmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \quad (\text{applying } R_1 = R_1 + R_2)$$

$$= a^2b^2c^2 2(1+1) = 4a^2b^2c^2,$$

Example 9. Show that

$$\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = (x+y+z)(x-z)^2.$$

Solution:

$$\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = \begin{vmatrix} 2(x+y+z) & x+y+z & x+y+z \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$$

operating $R_1 = R_1 + R_2 + R_3$

$$= (x+y+z) \begin{vmatrix} 2 & 1 & 1 \\ z+x & z & x \\ x+y & y & z \end{vmatrix}, \text{ (taking } (x+y+z) \text{ common from } R_1)$$

$$= (x+y+z) \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x-z & z-x & x \end{vmatrix}, \quad \left(\begin{matrix} \text{operating } C_1 = C_1 - 2C_2 \\ \text{and } C_2 = C_2 - C_3 \end{matrix}\right)$$

$$= (x+y+z) \begin{vmatrix} x-z & z-x \\ x-y & y-z \end{vmatrix}$$

$$= (x+y+z)(x-z) \begin{vmatrix} 1 & -1 \\ x-y & y-z \end{vmatrix}, \text{ (taking } (x-z) \text{ common from } R_1)$$

$$= (x+y+z)(x-z)(y-z+x-y)$$

$$= (x+y+z)(x-z)^2.$$

2.3 TEST YOUR UNDERSTANDING (A)

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1. Evaluate $\begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix}$.

2. Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$.

3. Evaluate $\begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix}$.

4. Show that

(a) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$ (b) $\begin{vmatrix} 1 & bc & a(b=c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0.$

5. Prove that

$$\begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix} = -(b-c)(c-a)(a-b)(a^2+b^2+c^2).$$

6. Show that

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$$

7. Prove that (without expanding)

$$\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ & b & 1+c \end{vmatrix} = 1 + a + b + c\ a$$

8. Show that

$$\begin{vmatrix} a-b-c & 2b & 2c \\ 2a & b-c-a & 2c \\ 2a & 2b & c-a-b \end{vmatrix} = (a+b+c)^3$$

9. Prove that

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

10. Show that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Answers

1. 21, **2.** xy **3.** $a^3+b^3+c^3-3abc$

2.4 MINORS

Consider the determinant of a square matrix A,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

When we delete any one row and any one column of $|A|$, we get a 2×2 determinant. For example, if we strike off the row and column passing through a_{11}, i. *e*., the first row and first column, we get the determinant as

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

This determinant is called the minor of the element a_{11} in determinant A.

Thus, the minor of the element in the determinant of the square matrix may be defined as a determinant which is left after deleting the row and column in which the element lies. The number of minors in a determinant will be equal to the number if elements therein. The following is the list of all nine minors in $|A|$.

The minors of a_{11}, a_{12} and a_{13} are

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \text{ and } \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \text{ respectively.}$$

The minors of a_{21}, a_{22} and a_{23} are

$$\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \text{ and } \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \text{ respectively.}$$

The minors of a_{31}, a_{32} and a_{33} are

$$\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \text{ and } \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \text{ respectively.}$$

In general, the determinant obtain by striking off the ith row and jth column of a matrix $A = [a_{ij}]_{n \times n}$ is called the minor of a_{ij} in $|A|$**.** The minor of element a_{ij} is determinant by M_{ij}.

2.5 CO-FACTORS OF A MATRIX

If we multiply the minor of the element in the ith row and jth column of the determinant of the matrix by $(-1)^{i+j}$ the product is called the co-factor of the element. It is usual to denote the co-factor of an element by the corresponding capital letter. Symbolically $A_{ij} = (-1)^{i+j} \times$ minor of a_{ij} in A

$$= (-1)^{i+j} M_{ij}.$$

If we consider a determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

We get

$$A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix},$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \text{ and so on.}$$

Note :

1. The sum of the products of the elements of any row (column) of a determinant with the corresponding co-factors is equal to the value of the determinant, i. *e.*,

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13},$$

$$= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}, \text{ etc.}$$

2. The sum of the products of the elements of any row (column) with the co-factors of the corresponding elements of any other row (column) is zero, i. *e*.,

$$0 = a_{11}A_{12} + a_{12}A_{22} + a_{13}A_{23},$$
$$= a_{13}A_{11} + a_{23}A_{21} + a_{33}A_{31}, \text{ etc.}$$

Example 10. ***Find theminors and co-factors of matrix*** A ***and use it to evaluate the determinant of the matrix***

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & \\ 1 & 4 & 8 \end{bmatrix}$$

Solution: The minors are calculated as follows:

As M_{ij} = minor of a_{ij}

$\therefore$ $$M_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 40 - 24 = 16$$

$$M_{12} = \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} = 16 - 6 = 10$$

$$M_{13} = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 8 - 5 = 3$$

$$M_{21} = \begin{vmatrix} 1 & -4 \\ 4 & 8 \end{vmatrix} = 8 + 16 = 24$$

$$M_{22} = \begin{vmatrix} 3 & -4 \\ 1 & 8 \end{vmatrix} = 24 + 4 = 28$$

$$M_{23} = \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} = 12 - 1 = 11$$

$$M_{31} = \begin{vmatrix} 1 & -4 \\ 5 & 6 \end{vmatrix} = 6 + 20 = 26$$

$$M_{32} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 18 + 8 = 26$$

$$M_{33} = \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} = 15 - 2 = 13$$

Now as co-factor A_{ij} of element $a_{ij} = (-1)^{i+j}M_{ij}$ $\therefore$

Co-factors

$$A_{11} = (-1)^2 M_{11} = +(16) = 16$$

$$A_{12} = (-1)^3 M_{12} = -(10) = -10 \; A_{13}$$

$$= (-1)^4 M_{13} = +(3) = 3$$

$$A_{21} = (-1)^3 M_{21} = -(24) = -24$$

$$A_{22} = (-1)^4 M_{22} = +(28) = 28$$

$$A_{23} = (-1)^5 M_{23} = -(11) = -11$$

$$A_{31} = (-1)^4 M_{31} = +(26) = 26$$

$$A_{32} = (-1)^3 M_{32} = -(26) = -26$$

$$A_{33} = (-1)^6 M_{33} = +(13) = 13$$

Also $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$$= 3(16) + 1(-10) - 4(3)\cdot$$

$$= 48 - 10 - 12 = 26$$

2.6 ADJOINT OF A MATRIX

Let $A = [a_{ij}]$ be a square matrix of order n and let A_{ij} denote the co-factor of a_{ij} in $|A|$. Then the adjoint of matrix $A$is defined as the transpose of the matrix $[A_{ij}]$ and is expressed by writing $adj\ A$.

Thus, in order to find the adjoint of a matrix, replace each element of this matrix by the co-factor of that element and then transpose this matrix of co-factors. The two operations may also be performed in reverse order, i. *e.,* first find the transpose of the given matrix and then replace each element by its co-factor.

Hence, if

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

we have

$$adj\ A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

where capital letters denote the co-factors of small letters in $|A|$.

i. *e.*, A_{ij} = co-factor of a_{ij} in $|A|$.

Some Properties

(i) If A is a square matrix,

$$A.\,(adjA) = (adjA).\,A$$
$$= |A|I$$

Where I is the matrix of the same order as A**.**

(ii) If $A = [a_{ij}]$ is a square matrix of order n, we have

$$|adj\ A| = |A|^{n-1}, \text{ provided } |A| \neq 0.$$

(iii) If A and B are two square matrices of $n \times n$ order each, we have

$$adj(AB) = (adj\ B).\,(adj\ A).$$

2.7 INVERSE OF A MATRIX

If, for a given square matrix A of order $\times\ n$, there exists of matrix B such that $AB = BA = I_n$ (where I_n is a unit matrix of order n), the square matrix B is said to be an inverse of A**.** We write B as A^{-1}, read as 'A inverse'. A matrix having an inverse is called an inverse matrix.

From the definition given above, we see that we can talk of an inverse of a square matrix only. Also, we find that, if B be an inverse of a square matrix only. Also, we find that, if B be an inverse of A**,** A is also an inverse of B**.**

We already know that

$$A.\,(adj\ A) = |A|I$$

or $\quad A\left(\frac{adj\ A}{|A|}\right) = I$**,** provided $|A| \neq 0$.

Hence, $\quad A^{-1}\left(\frac{adj\ A}{|A|}\right)$, if $|A| \neq 0$.

Thus, we find another from the inverse or reciprocal of the matrix A, which is quite suggestive of the procedure for finding an inverse.

A^{-1} is said to be the inverse of A because it possesses the property $AA^{-1} = A^{-1}A = I$**.**

Theorem 1. *The inverse of a matrix is unique.*

Proof. If possible, let B and C be two inverse of a square matrix A**.** Since B is an inverse of A, we have

$$AB = BA = I \quad \ldots(i)$$

Again, since C is an inverse of A, we have

$$AC = CA = I \quad \ldots\text{(ii)}$$

From (i), we have

$C(\ B) = CI = C$ …(iii) Also, from (ii), we have

$$(CA)B = IB = B \quad \ldots\text{(iv)}$$

We know that $C(AB) = (CA)B$.

Therefore, from (iii) and (iv) it follows that

$$B = C$$

i. *e*., the inverse is unique.

Theorem 2. *A square matrix A has an inverse, if and only if* $|A| \neq 0$.

OR

A square matrix A*is invertible, if and only if* A *is non-singular*.

Proof. The condition is necessary Let B

be the inverse of the matrix A**,** then

$$AB = I.$$

Therefore, $|A||B| = |I| = 1$.

Hence, $|A| \neq 0$.

The condition is sufficient. Suppose $|A| \neq 0$. Let us assume that

$B = \frac{adj\ A}{|A|}$

$AB = A.\left(\frac{adj\ A}{|A|}\right)$

$= \frac{1}{|A|}(A.adj\ A) = \frac{|A|I}{|A|} = I$

$BA = I$

$AB = BA = I$.

∴

Similarly,

∴

Hence, Ahas an inverse.

Theorem 3. Reversal Law : *If* A*and* B *are invertible square matrices of the same order, then* AB *is also invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

Given A, B are invertible matrices of same order, hence

$$|A| \neq 0, \quad \text{and} \quad |B| \neq 0.$$

$\Rightarrow |A||B| \neq 0 \Rightarrow |AB| \neq 0 \Rightarrow AB$ is invertible.

Let $(AB)^{-1} = C$, then $(AB)C = I = C(AB)$

Now, $\quad (AB)C = I \quad \Rightarrow \quad A(BC) = I$

$\Rightarrow \quad A^{-1}[A(BC)] = A^{-1}$

$\Rightarrow \quad (A^{-1}A)(BC) = A^{-1} \quad \Rightarrow \quad I(BC) = A^{-1}$

$\Rightarrow \quad BC = A^{-1} \quad \Rightarrow \quad B^{-1}(BC) = B^{-1}A^{-1}$

$\Rightarrow \quad (B^{-1}B)C = B^{-1}A^{-1} \quad \Rightarrow \quad IC = B^{-1}A^{-1}$

$\Rightarrow \quad C = B^{-1}A^{-1}$

Hence, $\quad (AB)^{-1} = B^{-1}A^{-1}$.

Example 11. ***Find the adjoint of the matrix*** $A = \begin{bmatrix} -2 & 3 \\ -5 & 4 \end{bmatrix}$

Solution : $\quad A = \begin{bmatrix} -2 & 3 \\ -5 & 4 \end{bmatrix}$

Here, co-factor of $a_{11} = (-1)^{1+1} 4 = 4$

Co-factor of $a_{12} = (-1)^{1+2}(-5) = -(-5) = 5$

Co-factor of $a_{21} = (-1)^{2+1} 3 = -3$

Co-factor of $a_{22} = (-1)^{2+2}(-2) = -2$

Adjoint of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -3 & -2 \end{bmatrix}' = \begin{bmatrix} 4 & -3 \\ 5 & -2 \end{bmatrix}$

Example 12. ***Find adjoint of*** $A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 3 \\ 4 & 1 & 5 \end{bmatrix}$

Solution : $\quad A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 3 \\ 4 & 1 & 5 \end{bmatrix}$

Now, If A_{ij} = co-factor of a_{ij}

Then $Adj\ A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$

$$Adj\,A = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix}1 & 3\\ 1 & 5\end{vmatrix} & (-1)^{1+2}\begin{vmatrix}3 & 3\\ 4 & 5\end{vmatrix} & (-1)^{1+3}\begin{vmatrix}3 & 1\\ 4 & 1\end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix}1 & 4\\ 1 & 5\end{vmatrix} & (-1)^{2+2}\begin{vmatrix}2 & 4\\ 4 & 5\end{vmatrix} & (-1)^{2+3}\begin{vmatrix}2 & 1\\ 4 & 1\end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix}1 & 4\\ 1 & 3\end{vmatrix} & (-1)^{3+2}\begin{vmatrix}2 & 4\\ 3 & 3\end{vmatrix} & (-1)^{3+3}\begin{vmatrix}2 & 1\\ 3 & 1\end{vmatrix} \end{bmatrix}'$$

$$Adj\,A = \begin{bmatrix} (5-3) & -(15-12) & (3-4) \\ -(5-4) & (10-16) & -(2-4) \\ (3-4) & -(6-12) & (2-3) \end{bmatrix}$$

$$Adj\,A = \begin{bmatrix} 2 & -3 & -1 \\ -1 & -6 & 2 \\ -1 & 6 & -1 \end{bmatrix}' = \begin{bmatrix} 2 & -1 & -1 \\ -3 & -6 & 6 \\ -1 & 2 & -1 \end{bmatrix}$$

Example 13. ***Find*** A^{-1} ***for*** $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$**.**

Solution : $A^{-1} = \dfrac{adj\,A}{|A|}$

The adjoint of A,as calculated in example 1 of the proceeding section, is as under :

$$adj\,A = \begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$$

$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = 1(1) + 2(1)$, expanding by the first row

$$= 3$$

$$\therefore \quad A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{5}{3} & \frac{4}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

Note. The answer can be verified from the fact that $A^{-1}A = I$, as shown below.

$$A^{-1}A = \begin{bmatrix} \frac{1}{3} & \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{5}{3} & \frac{4}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}+\frac{8}{3}-1 & \frac{4}{3}-\frac{4}{3} & \frac{2}{3}-\frac{2}{3} \\ -\frac{2}{3}-\frac{10}{3}+4 & -\frac{5}{3}+\frac{8}{3} & -\frac{4}{3}+\frac{4}{3} \\ \frac{1}{3}-\frac{4}{3}+1 & -\frac{2}{3}+\frac{2}{3} & \frac{2}{3}+\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Example 14. ***Calculate the inverse of the matrix***

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Solution : Here $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

Now, $|A| = 0$, if $ad - bc = 0$.

Let us assume that $ad - bc \neq 0$ so that A^{-1} exists.

But $A^{-1} = \frac{adj\, A}{|A|}$.

Now, we have

$$A_{11} = d, A_{12} = -c, A_{21} = -b \text{ and } A_{22} = a.$$

$$adj\, A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

∴

And

Example 15. ***Find the inverse of the matrix :***

$$\begin{bmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{bmatrix}$$

Solution : Let $A = \begin{bmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{bmatrix}$

$$|A| = 3(-16+8) + 10(4-4) - 1(8-16)$$

$$= -24 + 0 + 8 = -16 \neq 0 \quad \therefore A^{-1} \text{ exists}$$

The co-factors are

$A_{11} = -16 + 8 = -8,$

$A_{12} = -(4-4) = 0$'

$A_{13} = 8 - 16 = -8,$

$A_{21} = -(20-4) = -16$'

$A_{22} = -(6+2) = -4,$

$A_{23} = -(-12+20) = -8$'

$A_{31} = -20 + 8 = -12$

$A_{32} = -(6-2) = -4$

$A_{33} = 24 - 20 = 4$

$$\therefore \text{Matrix of co-factors} = \begin{bmatrix} -8 & 0 & -8 \\ -16 & -4 & -8 \\ -12 & -4 & 4 \end{bmatrix}$$

$$\therefore adj\, A = \begin{bmatrix} -8 & -16 & -12 \\ 0 & -4 & -4 \\ -8 & -8 & 4 \end{bmatrix} = -4 \begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(adj\, A) = \frac{4}{16}\begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$$

Then

Example 16. ***Given the matrix***

$$X = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}, \qquad \textbf{\textit{show that}}\, XX^{-1} = I_3.$$

Solution : We know that

$$X^{-1} = \frac{Adj}{|X|}.$$

We have $|X| = 1(4-3) - 4(-2-1) + 2(-3-2) = 3$

Let us find adjoint of X

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \end{vmatrix} = 1,$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = 3, \qquad A_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} = -5,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} = -2,$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1,$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = 0, \qquad A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} = 6$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = -3,$$

Hence, co-factor matrix $= \begin{bmatrix} 1 & 3 & -5 \\ -2 & 0 & 1 \\ 0 & -3 & 6 \end{bmatrix}$

$\therefore$ $$Adj\,\mathrm{X} = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 0 & -3 \\ -5 & 1 & 6 \end{bmatrix}$$

$\therefore$ $$\mathrm{X}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 & 0 \\ 3 & 0 & -3 \\ -5 & 1 & 6 \end{bmatrix}$$

$\therefore$ $$\mathrm{XX}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 4 & 2 \\ -1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 3 & 0 & -3 \\ -5 & 1 & 6 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1\times 1 + 4\times 3 + 2\times -5 & 1\times -2 + 4\times 0 + 2\times 1 & 1\times 0 + 4\times -3 + 2\times 6 \\ -1\times 1 + 2\times 3 + 1\times -5 & -1\times -2 + 2\times 0 + 1\times 1 & -1\times 0 + 2\times -3 + 1\times 6 \\ 1\times 1 + 3\times 3 + 3\times -5 & 1\times -2 + 3\times 0 + 2\times 1 & 1\times 0 + 3\times -3 + 2\times 6 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

Example 17. ***Show that*** $A = [2 \quad 1 \quad 2]$ ***satisfies the equation*** $A^2 - 4A - 5I = 0$ ***where*** I

$2 \quad 2 \quad 1$

is identity matrix and 0 ***denotes the zero matrix. Hence find the inverse of*** A.

$1 \quad 2 \quad 2$

$A = [2 \quad 1 \quad 2]$

$2 \quad 2 \quad 1$

$1 \quad 2 \quad 2$

$A^2 = [2 \quad 1 \quad 2] \times$

$2 \quad 2 \quad 1$

$1 + 4 + 4$

$= [2 + 2 + 4$

$2 + 4 + 2$

$4 \quad 8 \quad 8$

$4A = [8 \quad 4 \quad 8]$

$8 \quad 8 \quad 4$

$5 \quad 0 \quad 0$

$5I = [0 \quad 5 \quad 0]$

$0 \quad 0 \quad 5$

$9 \quad 8 \quad 8$

$4A - 5I = [8 \quad 9 \quad 8] -$

$8 \quad 8 \quad 9$

$9 - 4 - 5$

$= [\quad 8 - 8$

$8 - 8$

$0 \quad 0 \quad 0$

$= [0 \quad 0 \quad 0] =$

$0 \quad 0 \quad 0$

$1 \quad 2 \quad 2$

$|2 \quad 1 \quad 2| = 1(1 - 4) -$

$2 \quad 2 \quad 1$

Solution : Here,

1 2 2
[2
2
2 +
+
+

4
[8
8
8 −
9 −
8

0 =

2(2
∴ 1 2]

2 1
2 + 4 2 + 4 + 2 9 8 8
4 1 + 4 4 + 2 + 2] = [8 9 8]
4 2 + 2 4 + 4 + 1 8 8 9

Also,

and

8 8 5 0 0
∴ $A^2 - 4$ 8] − [0 5 0]
8 4 0 0 5

8 8 − 8
4 − 5 8 − 8]
− 8 9 − 4 − 5

$R.H.S.$

To find A^{-1}.

$$|A| = -2) + 2(4-2) = 5 \neq 0$$

Hence, A^{-1} exists.
Multiplying the both sides of the equation

$$A^2 - 4A - 5I = 0$$

by A^{-1}, we have

$$A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I = 0$$

or $A - 4I - 5A^{-1} = 0$ or ∴

$$5A^{-1} = A - 4I$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}.$$

2.8 TEST YOUR UNDERSTANDING (B)

5. For the matrices $A = [^1 \quad {}^{-2}]$ and $B = [\ ^5 \quad {}^6]$ prove that $(AB)^{-1} = B^{-1}A^{-1}$.

3 4 −3 2

4 3 3

6. If $A = [-1 \quad 0 \quad -1]$,

−4 −4 −3

Find $adj\ A$, and hence, find the inverse of A.

7. Verify the theorem $A.(adj\ A) = (adj\ A).A = |A|I_3$ for the matrix,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}.$$

8. If $A = \begin{bmatrix} 3 & 4 & 6 \\ 1 & 2 & 3 \\ 8 & 5 & 10 \end{bmatrix}$, verify that $A^{-1}A = I$.

9. If $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}$, find $(AB)^{-1}$.

10. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, show that $A^3 - 23A - 40I = 0$, where I is a unit matrix of order 3 and 0 is null matrix of order 3. Hence (not otherwise), compute A^{-1}.

11. Verify $A^2 + 3A + 4I = 0$ for the matrix

$$A = \begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}.$$

Hence, obtain the inverse of A.

12. Show that the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Satisfies the equation $A^3 - 6A^2 + 9A - 4I = 0$, and hence, deduce A^{-1}.

Answers

1. $adj\ A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & -3 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & 3 \end{bmatrix}$;

5. $\frac{1}{19}\begin{bmatrix} 16 & 12 & 1 \\ 21 & 11 & -7 \\ 10 & -2 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{40}\begin{bmatrix} -4 & 4 & 8 \\ 1 & 11 & 8 \\ 14 & 6 & 8 \end{bmatrix}$

6.

8. $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

2.9 SOLVING LINEAR EQUATIONS BY MATRIX INVERSE METHOD

With the help of matrices a set of linear equations can be solved to find unknown values. Suppose we have the following set of linear equations:

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$ $a_{21}x_1$

$+ a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$

:

:

$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nm}x_n = b_n$.

These equations may be written in the following way:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}; \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \mathrm{X}$$

If A is a non-singular matrix then

$$\mathrm{X} = A^{-1}B, \quad A^{-1} = \frac{Adj.\ A}{|A|}$$

or $$\mathrm{X} = \frac{Adj.\ A \cdot B}{|A|}$$

Criterion of Consistency:

Let $A\mathrm{X} = B$ be a system of n-linear equations in n unknowns

Case I : If $|A| \neq 0$, then the system is consistent and has a unique solution given by $\mathrm{X} = A^{-1}B$.

Case II : If $|A| = 0$ and $(adj\ A).B = 0$, then the system is consistent and has infinite many solutions. **Case III :** If $|A| = 0$ and $(adj\ A).B \neq 0$, then the system is consistent.

Note : If $|A| = 0$, then system will have infinite numbers of solutions.

Example 18. ***Solve the system of equations***

$$x + y + z = 6$$
$$x - y + z = 2$$
$$2x + y - z =$$

Solution: The system of equations in matrix from is $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$X = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Now

∴

∴ $x = 1, y = 2, z = 3$

Example 19. ***Solve the following equations by matrix inverse method***

$$x_1 + 2x_2 = 10$$
$$2x_1 - x_2 = 15$$

Solution: $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, B = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$

$$AX = B$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5$$

∴ $|A| = -5$

Co-factor of $1 = -1$,

Co-factor of $2 = -2$

Co-factor of 2 = −2

Co-factor of 1 = −1

∴ Matrix of co-factors= $\begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}$

$$Adj. = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}$$

$A^{-1} = \frac{1}{-5}\begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}$ or $A^{-1}B = -\frac{1}{-5}\begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 10 \\ 15 \end{bmatrix}$

$= \frac{-1}{5}\begin{bmatrix} -10 & +(-30) \\ -20 & +(-15) \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -40 \\ -35 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$

∴ $x_1 = 8, x_2 = 7$

2.10 TEST YOUR UNDERSTANDING (C)

1. Solve the following equations by matrix method :

$$x - y = 5; \; 3x + 4y = 7$$

2. Solve the following equations by Matrix method :

$$x + 3y + z = 16; \quad 2x + y + 3z = 19; \quad x + 2y + 4z = 25$$

3. Solve the following equations by using matrix method :

$$x + y + z = 6; \quad x + 2y + 3z = 14; \quad x + y + 7z = 24$$

4. Solve the following system of linear equations by using matrix method :

$$x - 2y = 3; \quad 6x - 3y = 5$$

5. Solve the following equations by matrix method

$$x_1 + 5x_2 = 11; \quad 4x_1 - 3x_2 = 9$$

6. Solve the following equations by inverse method

$$x_1 + x_2 + x_3 = 1; \quad 2x_1 + 2x_2 + 3x_3 = 6; \; x_1 + 4x_2 + 9x_3 = 3$$

7. Solve the following equations by inverse method

$$x_1 - 2x_2 + 5x_3 = 1; \quad 2x_1 - 4x_2 + 8x_3 = 2; \; -3x_1 + 6x_2 + 7x_3 = 1$$

8. A salesman has following record of sales during three months for three items A, B and C

which have different rates of commission.

Months	Sale of Units			Total Commission Drawn (in Rs.)
	A	B	C	
January	90	100	120	800
February	130	50	40	900
March	60	100	30	850

Find out rates of commission per item on A, B and C by matrix method.

Answers

1. $\frac{27}{11}, \frac{-1}{11}$ **2.** 1, 2, 5 **3.** 1, 2, 3 **4.** Inconsistent

5. $x_1 = 3, x_2 = 1$ **6.** $x_1 = 7, x_2 = -10, x_3 = 4$

7. $x_1 = \frac{-14}{19}, x_2 = \frac{-7}{38}, x_3 = \frac{4}{19}$ **8.** $x = 2, y = 4, z = 11$

2.11 SOLVING THE EQUATIONS BY CRAMER RULE

It is a method that adopts determinants therefore it is also known as solving equations by the method of determinant. By using this method, in a set of three (say) linear equations, the unknown values x, y and z can be obtained. To solve the equations, 4 determinants are to be computed. They are :

$|D| =$ For the information given in the matrix i. e. determinants of coefficient of x, y, z.

$|D_1| =$ Determinant obtained from $|D|$ by replacing its first column by the values of B.

$|D_2| =$ Determinant obtained from $|D|$ by replacing its second column by the values of B . $|D_3|$

$=$ Determinant obtained from $|D|$ by replacing its third column by the values of B.

(i) If $|D| \neq 0$, then the values of unknowns are given by : $x = \frac{|D_1|}{|D|}, y = \frac{|D_2|}{|D|}$ and $z = \frac{|D_3|}{|D|}$

(ii) If $|D| = |D_1| = |D_2| = |D_3| = 0$ then the given set of equations is consistent having infinite solutions.

(iii) If $|D| = 0$ and at least one of the determinants $|D_1|$, $|D_2|$ and $|D_3|$ is non-zero, then the system is inconsistent.

Example 20. ***Solve the following equations by Cramer's Rule :***

$$2x - y = 5$$
$$x - 4y = -1$$

Solution: The above equations can be presented in the form of matrices :

$$\begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 5 \\ -1 \end{vmatrix}$$

$$AX = B$$

Consider $|D| = \begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix} = (-8 + 1) = -7$

$$|D_1| = \begin{vmatrix} 5 & -1 \\ -1 & -4 \end{vmatrix} = (-20 - 1) = -21$$

$$|D_2| = \begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix} = -2 - 5 = -7$$

$$x = \frac{|D_1|}{|D|} = \frac{-21}{-7} = 3,\ y = \frac{|D_2|}{|D|} = \frac{-7}{-7} = 1$$

∴

∴ $x = 3, y = 1.$

Example 21. ***Solve the following equations***

$$6x + y - 3z = 5$$
$$x + 3y - 2z = 5$$
$$2x + y + 4z = 8$$

Solution: The above equations can be presented in the form of matrix

$$\begin{bmatrix} 6 & 1 & -3 \\ 1 & 3 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 8 \end{bmatrix}$$

$$AX = B$$

To solve the equations, consider the determinant

$$|D| = \begin{vmatrix} 6 & 1 & -3 \\ 1 & 3 & -2 \\ 2 & 1 & 4 \end{vmatrix} = 6(12 + 2) - 1(4 + 4) - 3(1 - 6)$$

$$= 6 \times 14 - 8 \times 1 + 5 \times 3 = 84 - 8 + 15 = 91$$

$$|D_1| = \begin{vmatrix} 5 & 1 & -3 \\ 5 & 3 & -2 \\ 8 & 1 & 4 \end{vmatrix} = 5(12+2) - 1(20+16) + 3(5-24)$$

$$= 70 - 36 + 57 = 91$$

$$|D_2| = \begin{vmatrix} 6 & 5 & -3 \\ 1 & 5 & -2 \\ 2 & 8 & 4 \end{vmatrix} = 6(20+16) - 5(4+4) - 3(8-10)$$

$$= 216 - 40 + 6 = 182$$

$$|D_3| = \begin{vmatrix} 6 & 1 & 5 \\ 1 & 3 & 5 \\ 2 & 1 & 8 \end{vmatrix} = 6(24-5) - 1(8-10) + 5(1-6)$$

$$= 114 + 2 - 25 = 91$$

$\therefore \quad x = \frac{|D_1|}{|D|} = \frac{91}{91} = 1$

$y = \frac{|D_2|}{|D|} = \frac{182}{91} = 2$

$z = \frac{|D_3|}{|D|} = \frac{91}{91} = 1$

$\therefore \quad x = 1, y = 2, z = 1.$

Example 22. ***The sum three numbers is 20. If we multiply first by 2 and add the second number and subtract the third, we get 23. If we multiply the first by 3 and add second and third to it, we get 46. Find the numbers.***

Solution: Let the number be x, y and z repectively. According, the question we have

$x + y + z = 20$ (First Condition)

$2x + y - z = 23$ (Second Condition)

$3x + y + z = 46$ (Third Condition)

The above equations can be presented in the form of a matrix.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 23 \\ 46 \end{bmatrix}$$

$AX = B$

To solve the equations, consider the determinant

$$|D| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix} = 1(1+1) - 1(2+3) + 1(2-3)$$

$$= 2 - 5 - 1 = 2 - 6 = -4$$

$$|D_1| = \begin{vmatrix} 20 & 1 & 1 \\ 23 & 1 & -1 \\ 46 & 1 & 1 \end{vmatrix} = 20(1+1) - 1(23+46) + 1(23-46)$$

$$= 40 - 69 - 23 = 40 - 92 = -52$$

$$|D_2| = \begin{vmatrix} 1 & 20 & 1 \\ 2 & 23 & -1 \\ 3 & 46 & 1 \end{vmatrix} = 1(23+46) - 20(2+3) + 1(92-69)$$

$$= 69 - 100 + 23 = 92 - 100 = -8$$

$$|D_3| = \begin{vmatrix} 1 & 1 & 20 \\ 2 & 1 & 23 \\ 3 & 1 & 46 \end{vmatrix} = 1(46-23) - 1(92-69) + 20(2-3)$$

$$= 23 - 23 - 20 = 0 - 20 = -20$$

$$\therefore \quad x = \frac{|D_1|}{|D|} = \frac{-52}{-4} = 13$$

$$y = \frac{|D_2|}{|D|} = \frac{-8}{-4} = 2$$

$$z = \frac{|D_3|}{|D|} = \frac{-20}{-4} = 5$$

Thus, $x = 13, y = 2$ and $z = 5$.

Example 23. ***Solve the following system of equations***

$$x + 3y + 4z = 8$$
$$2x + y + 2z = 5$$
$$5x + 2z = 7$$

Solution: $A = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 0 & 2 \end{vmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$

$$AX = B$$

$$|D| = 1(2-0) - 3(4-10) + 4(0-5) = 2 + 18 - 20 = 0$$

Since $|D| = 0$, the system may have infinite number of solutions

Now solve any two of the given equations. Solving the last two equations i. e. equation (ii) and (iii), we get

$$2x + y + 2k = 5,\ 5x + 2k = 7$$

$$\Rightarrow \quad x = \frac{7-2k}{5}$$

Putting this values of x in equation (ii) we get

$$y = 5 - 2x - 2k = 5 - 2\left(\frac{7-2k}{5}\right) - 2k$$

$$\Rightarrow \quad y = \frac{11-6k}{5}$$

Thus all the infinite solutions of the given system are given by

$$x = \frac{7-2k}{5},\ y = \frac{11-6k}{5},\ z = k$$

Where k is arbitrary real number. By putting different values of k, we can get different solutions. For example, putting $k = 1$, we get a particular solution $x = 1, y = 1, z = 1$.

2.12 TEST YOUR UNDERSTANDING (D)

1. Solve the following equations using Cramer's rule :

$$2x - y = 1; \qquad 7x - 2y = -7$$

2. By Cramer's rule, solve the following equations :

$$2x - y + z = 4; \qquad x + 3y + z = 12; \qquad 3x + 2y + z = 5$$

3. Solve the following linear equations by Cramer's rule :

$$x + y + z = 1; \quad x + 2y + z = 1; \qquad 3x - y - 2z = 6$$

4. Solve the following equations by Cramer's rule :

$$x + y + z = 9; \quad 2x + 6y + 7z = 55; \quad 2x + y - z = 0$$

5. Solve the following equations by Cramer's rule

$$2x - y + 3z = 9$$
$$x + 3y - z = 4$$
$$3x + 2y + z = 10$$

6. Solve the following system of equations

(a) $5x - 6y + 4z = 15$; $7x + 4y - 3z = 49$; $2x + y + 6z = 46$
(b) $x + y + z = 9$; $2x + 6y + 7z = 55$; $2x + y - z = 0$

7. Solve the following equations using Cramer's rule :

$$x - y + z = 4; \quad x + 3y + 2z = 12; \quad 3x + 2y + 3z = 10$$

8. Using determinants, show that the following system of equations has infinite number of solutions.

$$x + 3y + 4z = 8; \quad 2x + y + 2z = 5; \quad 5x + 2z = 7$$

9. Solve the following set of equations

$$x + y + z = 1$$
$$x + 2y + 3z = 4$$
$$x + 3y + 5z = 7$$

10. A manufacturer is manufacturing two types of products A and B. L_1 and L_2 are two machines which are used for manufacturing these two types of products. The time taken by both A and B on machines is given below:

	Machine L_1	Machine L_2
Product A	20 hrs.	10 hrs.
Product B	10 hrs.	20 hrs.

If 60 hours is the time available on each machine. Calculate the number of units of each type manufactured using matrix method.

Hint : $20x + 10y = 600$ $10x$
$+ 20y = 600$

Answers

1. $x = -3, y = -7$

2. $x = -\frac{20}{7}, y = \frac{9}{7}, z = 11$

3. $x = \frac{8}{14}, y = \frac{24}{14}, z = -\frac{42}{14}$

$x = 1, y = 3, z = 5$

4.

5. $x = 1, y = 2, z = 3$

6. **(a)**$x = 3, y = 4, z = 6$; **(b)**$x = 1, y = 3, z = 5$

7. System is inconsistent since $|D| = 0$ and $|D_1| \neq 0$

8. For $k = 1, y = 1, z = 1$

9. Infinite solutions, $x = k - 2, y = 3 - 2k, z = k$

10. $x = 20, y = 20$.

2.13 LET US SUM UP

- To every square matrix, we associate a single number that gives us an important information about that matrix. That number is called determinant of the matrix.
- The matrix which is not square does not possess determinant.
- Determinant is used to know whether the matrix is invertible or not.
- Determinant of any square matrix can be easily calculated by applying its various properties.
- Matrices and Determinants can be used to solve a system of linear equations.
- There are two methods to solve linear equations i.e. Method of Determinants (Crammer's Rule) and Matrix Inverse Method.

2.14 KEY TERMS

- **Determinant:** To every square matrix, we associate a single number that gives us an important information about that matrix. That number is called determinant of the matrix.
- **Minor:** The minor of the element in the determinant of the square matrix may be defined as a determinant which is left after deleting the row and column in which the element lies.
- **Adjoint of a Matrix:** A matrix obtained by interchanging the rows and columns of the co-factor matrix of the given square matrix.
- **Inverse of a Matrix**: The inverse of any square matrix 'A' is another square matrix 'B' such that AB = BA = I.

2.15 FURTHER READINGS

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B. COM (Hons.)

(Accounting and Taxation)
SEMESTER II

COURSE: BUSINESS MATHEMATICS AND STATISTICS

Unit–3: Functions

STRUCTURE

3.0 OBJECTIVES

After studying the Unit, students will be able to

- Define the Meaning of a Function.
- Understand the concept of Domain and Range of a Function.
- Differentiate between various types of Functions.
- Describe Algebra of Functions.
- Apply Functions in Business and Economics

Suppose A and B are any two non-empty sets. Let $A = \{a, b, c, d\}, B = \{x, y, z\}$. Suppose by some rule or other, we assign to each element of A, a unique element of B. Suppose a is associated to x, b is associated to y, c is associated to x and d is associated to z. The set of such assignment is called a '$function$' or '$mapping$' from A to B. If we denoted this set by f, then we write

$$f : A \rightarrow B$$

which is read as "f is a function af A to B" or "f is a mapping from A to B". In fact a function can be thought of like a machine: when we put x into the machine, it comes out with corresponding y .

Definition 1. *Let A and B be two given sets. Suppose they exists a rule denoted by f, which associates to each member of A, a unique member of B. Then f is called a function or a mapping of A to b. The mapping of A to B is denoted by* $f : A \rightarrow B$ *or by* $A \rightarrow B$.

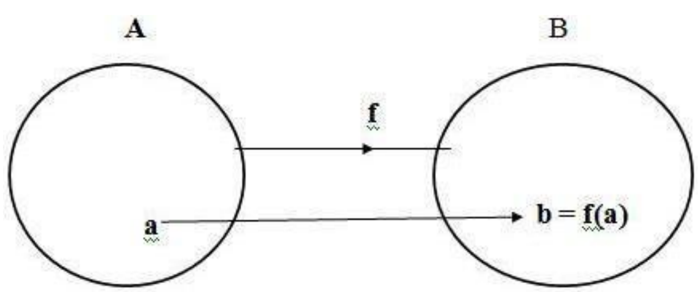


Fig. 1

Further,if $a \in A$, then the element in B which is assigned to a is called the f image of a or the value of the function f for a and is denoted by $f(a)$.

Definition 2. *Let x be some variable quantity. Then a rule that assigns a definite value to every value that x may have, is called a function of x.*

Definition 3. *If x and y represent members of two set A and B respectively, we say that y is a function of x and write* $y = f(\ )$, *provided we can associate, by some method or procedure, to every* $x \in A$, *a unique* $y \in B$.

(Note : The symbol '$\in$' stands for 'belongs to'**)**

Example. Giving the following two sets :

$A = \{$Tony Greig, Bobby Fischer, P. Padukone, Chris Evert$\}$ B
$= \{$Badminton, Tennis, Chess, Cricket, Hockey, Volley Ball$\}$.

Set A lists the players and B the games. For $x \in A$, and $y \in B$, we define a procedure for assigning a member set B to each member of A by x plays y. This rule thus defines a function from A to B that associates Tony Greig to Cricket, Bobby Fischer to Chess, P. Padukone to Badminton and Chris Evert to Tennis.

The general notation of a function is

$$y = f(x) \qquad \ldots(1)$$

It means y is a function of x, or $f(x)$ is the value of the function at x, or y is the image of x under f. In this notation, x is said to be the **independent variable,** since it may have any value from among the member of set A, while y is **dependent variable,** for the values of y depend upon the particular choice of x.

Note : The rule f should possess the characteristics that there may be some elements of the set B which are not associated to any element of the set A but each element of the set A must be associated to one and only one element of the set B. Two or more elements of the set A may be associated to the same element of the set B but association of one element A to more than one element in B is not permissible.

3.2 DOMAIN, CO-DOMAIN AND RANGE OF FUNCTION

Let f be a mapping of A into B. Then A is called the *domain* of the function f and B the *co-domain* of the function f. It is evident from the definition that each element of B need not appear as the image of an element in A. We define the range of f to consist of all those elements in which appear as f image of at least one element in A . There can be more than one element of A which have the same image in B. The image set $f[A]$ is called the range of f .

FUNCTIONS DEFINED AS SETS OF ORDERED PAIRS : Let A and B are any two non-empty sets, then a mapping f of A to B is a subset f of $A \times B$ satisfying the following conditions

:

(a) for each $a \in A$, $(a, b) \in f$, for some $b \in$ B; (b)
if $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$.

The first condition ensures that we have a rule that assigns to each element $a \in A$ some element $b \in B$. Thus each element in A will have image. The second condition guarantees that the image is unique. Accordingly, f is a function from A to B .

Note: If $f: A \to B$, it is important to distinguish between a function f and the value $f(x)$ of f for any element x. While f is a subset of $A \times B$, $f(x)$ is an element of the set B.

EQUAL FUNCTIONS : *Two functions f and g are said to be equal iff* :
(i) the domain of f *= domain of* g*,*
(ii) the co-domain of f *= the co-domain of* g *and (iii)* $f)(x) = g(x)$
for every x belonging to their common domain.

Example : *Let* $A = \{1,2\}$, $B = \{3,6\}$ *and* $f: A \to B$ *given by* $f(x) = x^2 + 2$ *and* $g: A \to B$ *given by* $g\ (x) = 3x$*. Then, we observe that* f *and g have the same domain and co-domain and also*

$$f(1) = 3 = g(1) \qquad and \qquad f(2) = 6 = g(2)\ Hence$$

$f = g$.

3.3 ALGEBRA OF FUNCTIONS

If f and g are two functions, we define

(i) The **sum function** $f + g$ by
$$(f + g)(x) = f(x) + g(x),$$
(ii) The **difference function** $f - g$ by
$$(f - g)(x) = f(x) - g(x),$$
(iii) The **product function** $f.\ g$ by

$(f. g)(x) = f(x). g(x)$, (iv) The quotient function f/g by where $g(x) \neq 0$, for any 'x' $(f/g)(x) = f(x)/g(x)$.

3.4 KINDS OF FUNCTIONS

If $f: A \to B$is a function, then f associates all elements of set A to elements in set B such that an element of set A is associated to a unique element of set B. Following these two conditions we may associate :

(i) difference elements of set A to different elements of set B, or

(ii) more than one element of set$A$ may be associated to the same element of set B, or

(iii) all elements in$B$ may have their pre-images in A.

Corresponding to each of these possibilities we define a type of a function as given below :

***(1)* ONE-ONE FUNCTION (INJECTION) :** *A function* $f: A \to B$ *is said to be a one-one function or an injection if different elements of A have different image in* B.

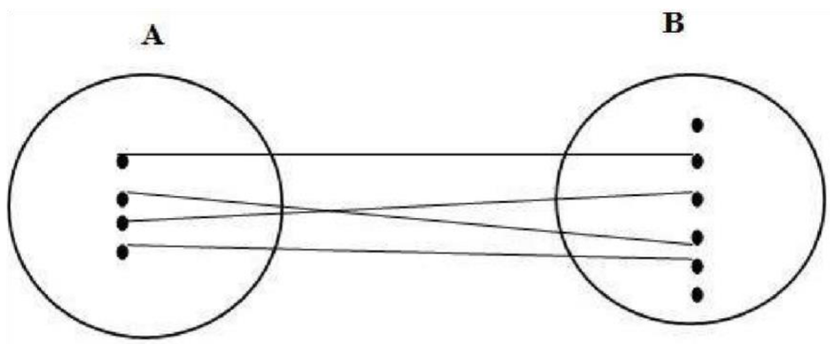


Fig. 2

Thus, $f: A \to B$ is one-one $\Leftrightarrow a \neq b$

$\Rightarrow \quad f(a) \neq f(b)$ for all $a, b \in A$.

$- f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$.

Examples :

(i) A function which associates to each country in the world its capital is one-one because different countries have their different capitals.

(ii) Let $X = \{1,2,3,4\}$; $Y = \{1,4,9,16\}$ *and* $f: X \to Y$ *s.t.* $f(x) = x^2\ \forall\ x \in X$, *then* f *is one- one mapping of X into Y, as no two distinct elements of X have the same f-image in Y.*

***(2)* MANY-ONE FUNCTION :** *A function* $f: A \to B$ *is said to be a many-one function of two or more elements of set A have the same image in B.*

Thus, $f: A \to B$ is many one function if there exist $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$.

In other words, $f: A \to B$ is a many-one function if it is not a one-one function :

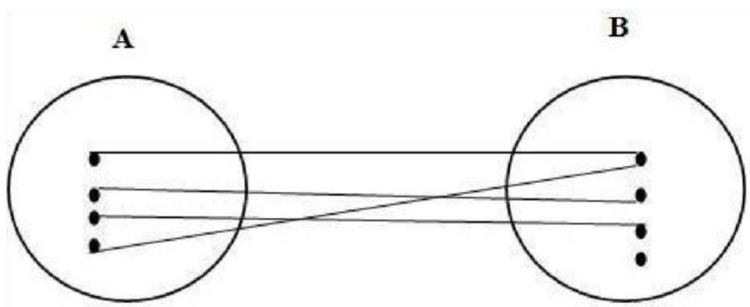


Fig. 3

Examples :

(i) $A = \{-1, 1, -2, 2\}$ *and* $B = \{1, 4, 9, 16\}$. *Consider* $f: A \to B$ *s.t.* $f(x) = x^2$. *Then* $f(-1) = 1, f(1) = 1, f(-2) = 4, f(2) = 4$.

Clearly 1 *and* -1 *have the same image. Similarly,* 2 *and* -2 *also have the same image. So,* f *is a many-one function.*

(ii) Consider a function $f: Z \to Z$ *given by* $f(x) = |x|$.

Then f *is a many-one function because for every* $a \in Z$, $a \neq 0$, $a \neq -a$ *but* $f(a) = f(-a)$.

$$[\because |a| = |-a|]$$

***(3)* ONTO FUNCTION (SURJECTION):** *A function* $f: A \to B$ *is said to be an onto function or a surjection if every element of B is the f-image of some element of* A, *i. e., if* $f(A) = B$ *or range of* f *is the co-domain of* f .

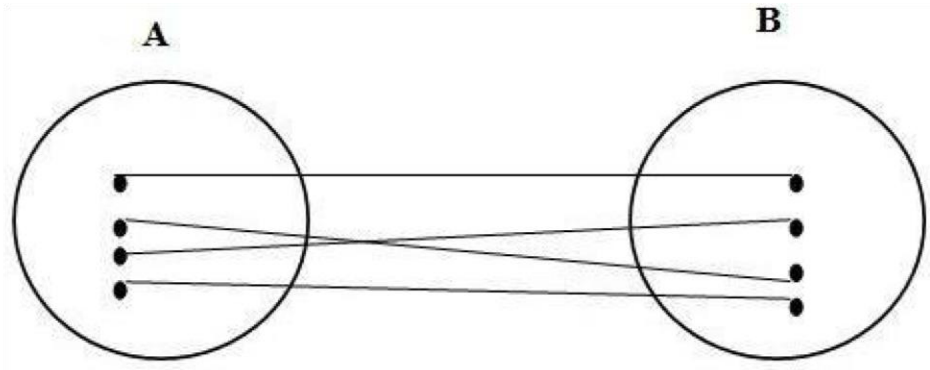


Fig. 4

Thus, $f: A \to B$ is a surjection iff for each $b \in B$, $\exists a \in A$ such that $f(a) = b$.

(4) **INTO FUNCTION:** *A function of* $f: A \to B$ *is an into function if there exists an element in B having no pre-image in* A .

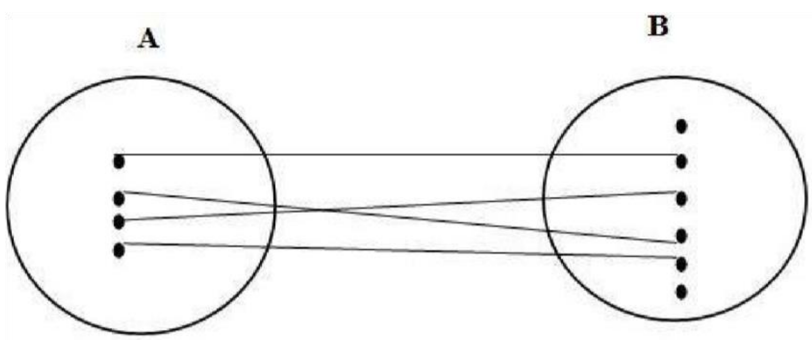


Fig. 5

In other words, $f: A \to B$ is into function if it is not an onto function.

Examples :

(i) *Let* $A = \{-1, 1, 2, -2\}$, $B = \{1, 4\}$ *and* $f: A \to B$ *be a function defined by* $f(x) = x^2$. *The* f *is onto because* $f(A) = \{f(-1), f(2), f(-2)\} = \{1, 4\} = B$

(ii) *A function* $f: N \to N$ *defined by* $f(x) = 2x$ *is not an onto function, because* $f(N) = \{2, 4, 6, \dots \dots\} \neq N$ *(co-domain).*

(5) **BIJECTION (ONE-ONE ONTO FUNCTION):** *A function* $f: A \to B$ *is a bijection if it is one-one as well as onto.*

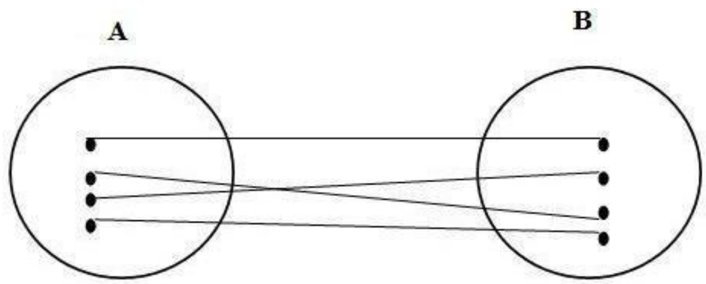


Fig. 6 *In other words, a function* $f: A \to B$ *is a bijection if*

:

(i) It is one-one, i. e., $f(x) = f(y) \Rightarrow x = y$ *for all* $x, y \in A$.
(ii) It is onto, i. *e.,* for all $y \in B$, *there exists* $x \in A$ *such that* $f(x) = y$.

Example : Let A *be the se of even integers and* B *be the set of odd integers, then the mapping* $f: A \to B$ *given by*

$$f(x) = x + 1, \forall\, x \in A$$

is one-one onto.

***(6)* ONE-ONE INTO MAPPING :** *Any mapping which is one-one as well as into is called one-one into mapping.*

Example : Let X *be the set of integers and* Y *be the set of all even integers, then the mapping* $f: X \to Y$. *s.t.* $f(x) = 2x$, $x \in X$ *is an into mapping which is also one-one.*

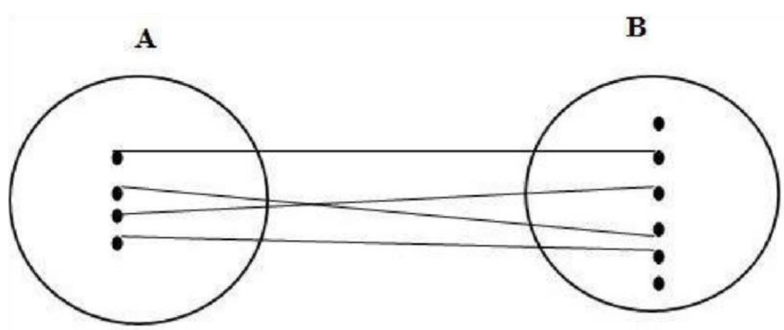


Fig. 7

Example 1. ***Given*** $f(\mathrm{x}) = 2\mathrm{x}^2 - 3\mathrm{x} + 1$***; find*** $f(2), f(0), f(-3)$**.**

Solution : $f(2) = 2 \,.\, 2^2 - 3 \,.\, 2 + 1 = 2 \,.\, 4 - 6 + 6 = 8 - 6 + 1 =$ 3;

$$f(0) = 2 \,.\, 0 - 3 \,.0 + 1 = 1;$$

$$f(-3) = 2(-3)^2 - 3(-3) + 1 =$$

$$18 + 9 + 1 = 28.$$

Example 2. ***The rule defining a function*** f ***is as follows :***
(i) $\frac{100}{\mathrm{x}}$ ***(ii)*** $\frac{\mathrm{x}}{\mathrm{x}-1}$
. Find the domain of definition of f **.**

Solution :

(i) $f(x) = \frac{100}{x}$, here $f(x)$ is undefined at $x = 0$. So the domain of definition of f is the set of real numbers except 0.

(ii) $f(x)$ is undefined for $x = 1$ only. Hence the domain of definition of f is the set of all real numbers except 1.

Example 3. ***Find the range of*** $f(\mathrm{x}) = \mathrm{x}^2$**,** $\mathrm{x} \in R$***(real number).***

Solution : $f(0) = 0$ and for every non-zero real values of x, (positive or negative) x^2 is always positive. So we get $0 \leq$ range $< \infty$.

Example 4. ***If*** $f(\mathrm{x}) = \mathrm{x} + |\mathrm{x}|$**,** ***find*** $f(3)$ ***and*** $f(-3)$ ***and show also they are not equal.***

Solution : $f(3) = 3 + |3| = 3 + 3 = 6$,

$f(-3) = -3 + |-3| = -3 + 3 = 0$, As $6 \neq 0$, so $f(3) \neq f(-3)$. [**Note :** If $f(x) = f(-x)$, in this case $f(3) = f(-3)$, then $f(x)$ will be an even function of x].

Example 5. ***Show that*** $f(\mathrm{x}) = \frac{\mathrm{x}^2-4}{\mathrm{x}-2}$ ***is undefined for*** $\mathrm{x} = 2$**.**

Solution : $f(x) = \frac{x^2-4}{x-2}$;putting $x = 1$,

$$f(1) = \frac{1-4}{1-2} = \frac{-3}{-1} = 3$$

$\because$ $f(1) = 3$, which is definite

Again for $x = -1, f(-1) = \frac{1-4}{-1-2} = \frac{-3}{-3} = 1$ so $f(x)$ is definite at $x = 1$

$\because \quad f(-1) = 1$, which is definite so this define at $x = -1$

For $x = 2$, we get $f(2) = \frac{4-4}{2-2} = \frac{0}{0}$

$\because \quad f(z) = \frac{0}{0}$, which is meaningless

$\therefore \quad f(x)$ cannot be defined for $x = 2$.

Example 6. ***Find the domain of the function*** $f(x) = \frac{x}{x^2-9}$.

Solution : Here $f(x)$ has a unique value except for $x = 3, -3$.

For $x = 3, f(3) = \frac{3}{9-9} = \frac{3}{0}$(undefined)

$= -3, f(-3) = \frac{-3}{9-9} = \frac{-3}{0}$(undefined) $\therefore$ domain of the function $f(x)$ is $-\infty < x < -3; -3 < x < 3; 3 < x < \infty$.

Example 7. ***Given the function :***

$$f(x) = 5^{-2x} - 1, \quad \textbf{\textit{for}} \quad -1 \le x < 0$$

$$= \frac{x^2-2}{x-2}, \quad \textbf{\textit{for}} \quad 0 \le x < 1.$$

$$= \frac{2x}{x^2-1}, \quad \textbf{\textit{for}} \quad 1 \le x \le 3.$$

Find $f(-1)$, $f(0)$, $f\left(\frac{1}{2}\right)$, $f(2)$.

Solution : Now $f(-1) = 5^{-2(-1)} - 1$ (since -1 lies in $-1 \le x < 0$)

$= 5^2 - 1 = 25 - 1 = 24$. Point

$0, \frac{1}{2}$ lies in the second interval, so

$f(0) = \frac{0-2}{0-2} = \frac{-2}{-2} = 1$, $f\left(\frac{1}{2}\right) = \frac{\frac{1}{4}-2}{\frac{1}{2}-2} = \frac{7}{6}$

$f(2) = \frac{2.2}{2^2-1} = \frac{4}{4-1} = \frac{4}{3}$ (as 2 lies in the 3rd interval)

Example 8. ***If*** $f(x) = x^2 - x$, ***prove that*** $f(h+1) = f(-h)$

Solution : $f(h+1) = (h+1)^2 - (h+1)$

$= h^2 + 2h + 1 - h - 1 = h^2 + h$

$f(-h) = (-h)^2 - (-h) = h^2 + h$

$\therefore \quad f(h + 1) = f(-h)$

Example 9. ***If*** $f(x) = \quad x^2 - 5x + 4$***, for what values of x is*** $2f(x) = f(2x)$**?**

Solution : $f(2x) = 2(2x)^2 - 5(2x) + 4$

$= 2 .4x^2 - 10x + 4 = 8x^2 - 10x + 4$

$2f(x) = 2(2x^2 - 5x + 4) = 4x^2 - 10x + 8$

By condition we have, $8x^2 - 10x + 4 = 4x^2 - 10x + 8$ or, $4x^2 = 4$ or, $x^2 = 1$ or, $x = \pm 1$.

Example 10. ***If*** $f(x) = x^3 - \frac{1}{x^3}$***, then show that*** $f(x) + f(\frac{1}{x}) = 0$.

Solution : We have :

$f(x) = x^3 - \frac{1}{x^3}$

$\therefore \quad f(\frac{1}{x}) = \left(\frac{1}{x}\right)^3 - \left(\frac{1}{1/x}\right)^3 = \frac{1}{x^3} - x^3$

$f(x) + f(\frac{1}{x}) = x^3 - \frac{1}{x^3} + \frac{1}{x^3} - x^3 = 0.$

$\therefore$

Example 11. ***If*** $f(x) = log\left(\frac{1+x}{1-x}\right)$***, show that*** $f(x) + f(y) = f\left(\frac{x+y}{1+xy}\right)$.

Solution : We have

$f(x) = log\left(\frac{1+x}{1-x}\right)$

$\therefore \quad f(y) = log\left(\frac{1+y}{1-y}\right)$

$f(x) + f(y) = log\left(\frac{1+x}{1-x}\right) + log\left(\frac{1+y}{1-y}\right)$

$= log\left(\frac{1+x}{1-x}\right)\left(\frac{1+y}{1-y}\right)$

$= log\left(\frac{1+x+y+xy}{1-x-y+xy}\right)$

$x+y \qquad 1+\left(\frac{x+y}{1+xy}\right)$

Then,

…(i)

Again, $f\left(\frac{}{1+xy}\right) = \log\left[\frac{}{1-\left(\frac{x+y}{1+xy}\right)}\right]$

$= log\left(\frac{1+xy+x+y}{1+xy-x-y}\right)$

…(ii)

From (i) and (ii) it is clear that $f(x) + f(y) = f\left(\frac{x+y}{1+xy}\right)$.

Example 12. ***Find the domain of each of the following real valued functions***

(i) $f(x) = \frac{1}{x+2}$ ***(ii)*** $f(x) = \frac{5x-4}{x^2-3x+2}$ ***(iii)*** $f(x) = \sqrt{x-4}$

Solution : (i) We have $f(x) = \frac{1}{x+2}$

Clearly, $f(x)$ assumes real values for all x except for the values of x satisfying $x + 2 = 0$, i. $e.$, $x = -2$.

Hence, domain of the function $f(x)$

i. $e.$, Domain $(f) = R - \{-2\}$

(ii) We have $f(x) = \frac{5x-4}{x^2-3x+2}$

Clearly, $f(x)$ assumes real values for all x except for the values of x for which $x^2 - 3x + 2 = 0$, i. $e.$, $x = 1, 2$.

∴ Domain $(f) = R - \{1, 2\}$

(iii) We have $f(x) = \sqrt{x-4}$

Clearly, $f(x)$ assumes real value, if $x - 4 \geq 0 \Rightarrow x \geq 4 \Rightarrow x \in (4, \infty)$

Hence, Domain $(f) = (4, \infty)$.

3.5 TEST YOUR UNDERSTANDING (A)

1. (i) If $f(x) = x^2 + 2x^4$ verify that $f(x) = f(-x)$, (ii) $f(x) = x + 2x^3$ verify that $f(-x) = -f(x)$

2. If $f(x) = (x-1)(x-2)(x-3)$ find the values of :

(i) $f(1)$ (ii) $f(2)$ (iii) $f(3)$ (iv) $f(0)$

[**Ans.** 0; 0; 0; -6]

3. If $y = 5$ for every values of x, can y be regarded as a function of x ? [**Ans.** Yes]

4. $f(x) = x + |x|$, are $f(4)$ and $f(-4)$ equal ?

[**Ans.** No]

5. If $(x) = b\frac{x-a}{b-a} + a\frac{x-b}{a-b}$, then $f(a) + f(b) = f(a+b)$.

6. Show that $\frac{x^2-6x+8}{x^2-8x+12}$ is undefined for $x = 2$ and also find $f(6)$.

[**Ans.** ∞]

7. Let $f(x) \quad = 2^{-x}, \; -1 \leq x < 0$

$= 4, \quad 0 \leq x < 1$

$= 2x - 1, \; 1 \leq \quad \leq 3.$

Calculate $f(-1), f(0), f(1), f(3)$.

[**Ans.** 2; 4; 1; 5]

8. If $f(x) = x^2 + x$ prove that $f(h+1) - f(h-2) = 6h$.

9. Find the domain of f , if

(i) $f(x) = \frac{1}{\sqrt{5-x}}$ (ii) $f(x) = \frac{x^2+3x+5}{x^2-5x+4}$

[**Ans.** Domain (f) = $(-\infty, 5)$] [**Ans.** Domain (f) = $R - \{1, 4\}$]

10. Find the domain of the function $f(x)$, defined by : $f(x) = \sqrt{4-x} + \frac{1}{\sqrt{x^2-1}}$

[**Ans.** Domain (f) = $(-\infty, -1) U (1, 4)$]

3.6 CLASSIFICATION OF REAL FUNCTIONS

(1) EXPLICIT AND IMPLICIT FUNCTIONS

In an explicit function, we have dependent variable on one side of the equation and the independent variable on the other. We have been dealing so far with this type of functions. Thus, an explicit function is represented by the form $y = f(x)$. In implicit functions, we have both the variables (i. e., dependent and independent ones) on the same side (i. e., left hand side or right hand side or both sides). The general form of an implicit function is $f(x, y) = C$, where C is a constant.

The following are the examples of explicit functions :

$y = x^2$, $y = e^2$ and $y = x^3 + 4x - e^x$ The

following are the examples of implicit functions :

$$x^2 + y^2 = 1,\ e^{x+y} = 2$$

and $\qquad 3x^2 + 4xy = 2x^3y^2 + 10x - 8y.$

(2) EVEN AND ODD FUNCTIONS

Ley$y = f(x)$ be a function with its domain of definition $D(f)$. If, for every $x \in D(f)$, $-x$ also belongs to $D(f)$, the function is said to be even or odd according as :

$$f(-x) = f(x) \text{ or } f(-x) = -f(x).$$

(3) LINEAR FUNCTION

A function of the form $f(x) = mx + c$ (where m and c are real numbers) is called a linear function. Its domain is set of real numbers. Replacing $f(x)$ by y, the function may be rewritten as

$$y = mx + c \qquad \ldots(1)$$

In this case, y is a dependent variable and x is an independent variable. Also, m is the slope and c is the intercept on y –axis. For example, $y = 3x + 4$ is a linear equation with slope 3 and intercept on y –axis 4. The equation (1) is also known as the equation of a straight line in slope-intercept form.

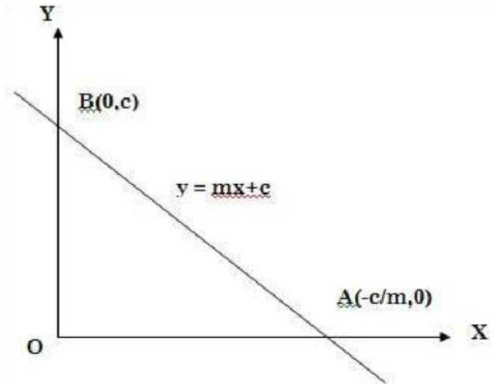


Fig. 8

The graph of the linear function $y = mx + c$ is a straight line. To draw a straight line, we needed to find any two points that lie on the line and then to join them by a straight line and extend the line further on both the direction.

(4) QUADRATIC FUNCTIONS AND PARABOLAS

A function of the form

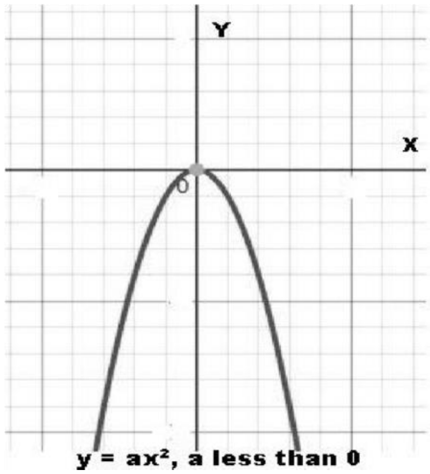
$$f(x) = ax^2 + bx + c \quad \ldots(1)$$

(where , b and c are the constants and $a \neq 0$) is called a **quadratic function**. The graph of quadratic function.

(1) is a curve, called a **parabola**. Its domain is the set of all real numbers. Putting $b = c = 0$ in (1) we get a simplest quadratic function

$$y = ax^2$$

Its graphs when a is negative and when it is positive are the parabolas as shown in Fig. and Fig. respectively. The lowest point on the graph when $a > 0$ occurs at the origin, while the origin is the highest point when $a < 0$. In either case, this point is called the **vertex** of the parabola.



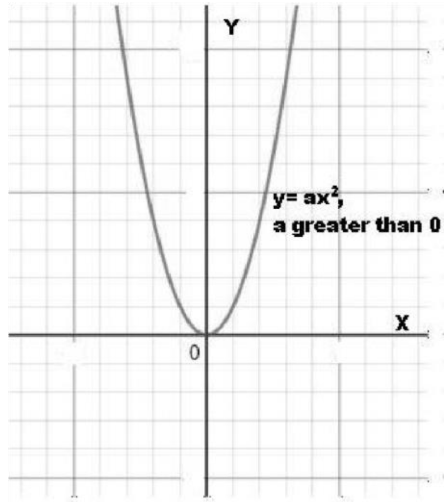


Fig. 9 **Fig. 10**

(5) POLYNOMIAL FUNCTION

A function of the form $f(x) = a_0x^4 + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots\ldots\ldots\ldots.. + a_n$, where $a_0,\ a_1,\ a_2 \ldots\ldots\ldots a_n$ are constants and n is a non-negative integer, $a_0 \neq 0$, is called a polynomial function of x of degree n.

(6) RATIONAL FUNCTION

A function of the form $(x) = \frac{p(x)}{q(x)},\ q(x) \neq 0$ where $p(x)$ and $q(x)$ are polynomials is

called a rational function. For example, $\frac{2x+3}{x^2+5}$ is a rational function.

(7) EXPONENTIAL FUNCTION

A function having a variable base and a constant exponent, $viz.,\ y = x,\ x^3,\ x^9,$ etc., is called a power function while *a function having a constant base and variable exponent is called an exponential function*. For example, $y = a^x$, $(a > 0, a \neq 1)$ is an exponential function, where a is the base and x is a real number exponent.

(8) LOGARITHMIC FUNCTIONS

If $a > 0$ and $a \neq 1$, the exponential function $x = a^y$ has an inverse function which is called the *logarithmic function*to the base a and is denoted by $y = log_a\, x$. Thus,

$$x = a^y \Rightarrow log_a\, x = y\ log_a\, a$$

⇒ $y = log_a x$, which is called logarithmic function.

Thus, if $x > 0$, the logarithmic function $log_a x$ is the exponent to which the base a must be raised to get x.

For example :

$2^3 = 8 \quad \therefore \quad log_2 8 = 3;$

(9) MONOTONIC FUNCTION

Let us consider any two numbers x_1 and x_2 in the domain of definition $[a, b]$ of $f(x)$, such that $x_2 > x_1$. Then $f(x)$ will be said to be *monotonic*.

(i) increasing in $[a, b]$ if $f(x_2) \geq f(x_1)$ (ii)
decreasing in $[a, b]$ if $f(x_2) \leq f(x_1)$.

(10) GREATEST INTEGRAL FUNCTION

$f(x) = [x] =$ greatest integer less than or equal to x. i.
e., if $0 \leq x < 1$ the function value is always 0.

if $1 \leq x < 2$, the function value is always 1. if $2 \leq x < 3$, the function value is always 2. again if $-1 \leq x < 0$, the functional value is always -1 and so on.

(11) ABSOLUTE VALUE FUNCTION OR MODULUS FUNCTION

A function $y = f(x) = |x|$ is called an *absolute value function (or modulus function).* It is defined as follows :

$$y = f(x) = x, \text{ for } x > 0$$

$$= 0, \text{ for } x = 0$$

$$= -x \text{ for } x < 0$$

Example : When $x = 4$, $y = |4| = 4$

$$= 0, y = |0| = 0$$

$$= -4, y = |-4| = 4$$

(12) IDENTITY FUNCTION

LetX be any set and the function f: $X \to X$ be defined by the formula $f(x) = x \ \forall \ x \in X$, i. *e.*, each element of X is mapped on itself, then f is called *the identity map or the identity function*on X.

We denote this function by I_x usually. Thus if I_x denotes the identity mapping on a set X, we have

$$I_x(x) = x, \forall \ x \in X.$$

Example : Let$A = \{a, b, c, d\}$. Then $f = \{(a, a), (b, b), (c, c), (d, d)\}$ is an identity mapping of A. Identity mapping is always one-one onto.

(13) INVERSE FUNCTION

Definition : Let f: $X \to Y$ be a bijection. Then a function g: $Y \to X$ which associates each element $y \in Y$ to a unique element $x \in X$ such that $f(x) = y$ is called the inverse of$f$, i. *e.*,

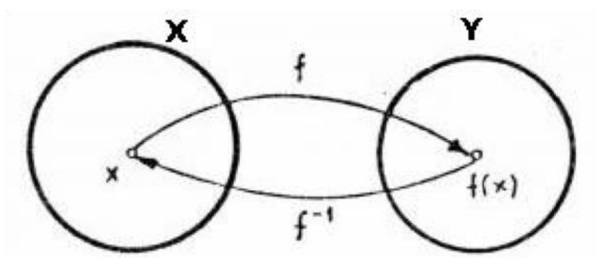


Fig. 11

$$f(x) = y \Leftrightarrow f^{-1}(y) = x.$$

The inverse of$f$ is generally denoted by f^{-1}.

Thus, if f: $X \to Y$ is a bijection, thenf^{-1}: $Y \to X$ is such that

$$f(x) = \Leftrightarrow f^{-1}(y) = x.$$

Example : If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and f: $A \to B$ is given by $f(x) = 2x$, then

$$f(1) = 2, f(2) = 4, f(3) = 6 \text{ and } f(4) = 8$$

So, $f = \{(1, 2), (2, 4), (3, 6), (4, 8)\}$
Which is clearly a bijection.

∴ $f^{-1} = \{(2, 1), (4, 2), (6, 3), (8, 4)\}$

(14) COMPOSITE FUNCTION

Let X, Y, Z be three sets and f be a function defined from X to Y and g be a function defined from Y to Z.

i. *e*., $f: X \to Y$ and $g: Y \to Z$.

By $f: X \to Y$ we mean that to every element $x \in X$, there corresponds a unique element $f(x) \in Y$.Since the domain of g is Y, so by the function $g: Y \to Z$we mean to every element $f(x) \in Y$there corresponds a unique element of $g[f(x)] \in Z$. Thus we notice that to every element $x \in X$ there corresponds a unique element $g[f(x)] \in Z$ under the mappings $f$and$g$. This implies that there exists a mapping from X to$Z$. This mapping is called the composite mapping or the product mapping of$f$ and g and is denoted by gof or gf.

Definition: Let $f: X \to Y$ and $g: Y \to Z$, then the composite of the functions f and g denoted by gof or$g\,f$ is mapping $gof: X \to Z$ $s.t.$ $(gof)(x) = g[f(x)], \forall\, x \in X$.

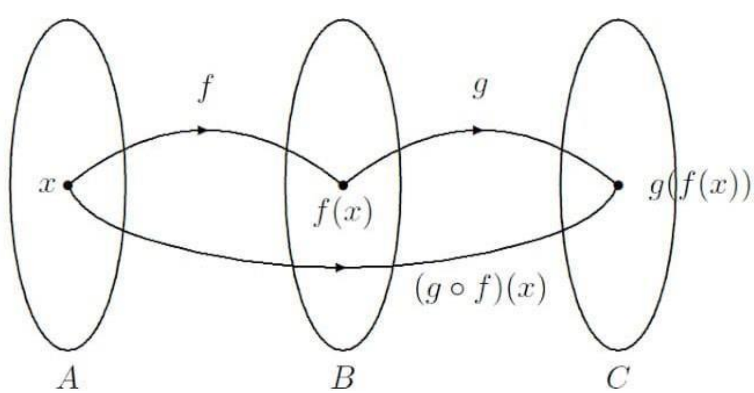


Fig. 12

Example13.*Let* f ***and*** g ***be two functions defined by*** $f(x) = \sqrt{x-1}$ ***and***$g(x) = \sqrt{(4-x^2)}$ ***,then describe the functions :***

(i)$(f-2g)(x)$ (ii)$(fog)(x)$

Solution : We have

$$f(x) = \sqrt{x-1}$$

$$g(x) = \sqrt{(4-x^2)}$$

$$(f-2g)(x) = f(x) - 2g(x)$$

$$= \sqrt{(x-1)} - 2\sqrt{(4-x^2)}$$

$$(fog)(x) = f[g(x)]$$

$$= f[\sqrt{(4-x^2)}]$$

$$= \sqrt{\{\sqrt{(4-x^2)} - 1\}}$$

And

(i)

(ii)

Example 14.*Let* $f: A \to B$ ***such that*** $(x) = x - 1$ ***and*** $g: B \to C$ ***such that*** $g(y) = y^2$**.** ***Find*** $fof(y)$**. Solution**

: We have

$$f: A \to B,\ g: B \to C\ f(x) = x - 1$$

and $g(y) = y^2$.

Now $fof(y) = f[f(y)]$

$$= f(y-1)$$

$$= (y-1) - 1 = y - 2.$$

Example 15. ***Let*** $f: N \to R\ s.t.\ f(x) = 2x - 3$ ***and*** $:x \to R\ s.t.\ g(x) = \frac{x-3}{2}$**.** ***Find the formula for*** $gof: N \to R$**. Solution**

: We have

$f(x) = 2x - 3$

and $g(x) = \frac{x-3}{2}$

$gof(x) = g[f(x)]$

$= g[2x - 3]$

$= \frac{2x-3-3}{2} = x - 3.$

16. ***If*** $f(x) = \frac{1}{1-x}$ ***and*** $(x) = \frac{x-1}{x}$

$\therefore$

Example , ***find the value of*** $g\,[f(x)]$**.**

Solution : We have

$$f(x) = \frac{1}{1-x}$$

$$g(x) = \frac{x-1}{x}$$

Now $g[f(x)] = g\,(\frac{1}{1-x})$

$$= \frac{\frac{1}{1-x}-1}{\frac{1}{1-x}}$$

$$= \frac{1-(1-x)}{1} = x$$

.

3.7 SPECIAL FUNCTIONS RELATED TO BUSINESS AND ECONOMICS

(1) DEMAND FUNCTION

The demand of a particular commodity depends on its price. If the price is less, the demand will be higher. Similarly, as the price increases the demand keeps on decreasing.

Thus, we can say that the demand of a commodity is a function of its price,

i. *e.,* $x = f(p)$

Where 'x' is the demand function and 'p' is the price of a commodity.

(2) SUPPLY FUNCTION

As the price increases, the supply tends to be increased by the producers. So, there is a direct relationship between supply and price of a commodity and mathematically, it can given as

$s = f(p)$ Where 's' is the supply function
and 'p' is the price of commodity.

(3) COST FUNCTION

The total cost at any level of output of any commodity bears a relationship with its quantity. Actually, the total cost does not increase in direct proportion to the number of units manufactured, but it is the sum of the fixed cost and the variable cost at any level of output, *i. e.,* Total Cost = Fixed Cost + Variable Cost Thus, cost function can be expressed as

$$C(x) = V(x) + F(x)$$

Where 'x' is the number of units produced.

(4) TOTAL REVENUE FUNCTION

It is the amount received by a firm by selling its product. Higher the sale, more would be the revenue of the firm.

So, mathematically $R = f(x)$

Where 'R' is the revenue function and 'x' is the number of units sold.

(5) PROFIT FUNCTION

Profit on a commodity depends upon its cost and sale price and its a function of number of units sold. Thus, mathematically

$$P(x) = R(x) - C(x)$$

Example 17. For a new product, a manufacturer sets up an infrastructure such that the manufacturing unit involves fixed cost of Rs. 50,000 and the additional cost, i. $e.$ variable cost for producing each unit is Rs. 150 per unit. The selling price of each unit would be Rs. 500. Write down the cost function $C(x)$, Revenue function $R(x)$ and profit function $CP(x)$ for x units of the product. Also determine the profit when 400 units are sold. **Solution :** Cost function

$$C(x) = 50{,}000 + 150x$$

Where 'x' is the number of units sold.

Also, Revenue function

$$R(x) = 500x$$

So, Profit function

$$P(x) = R(x) - C(x)$$

$$= 500x - (50,000 + 150x)$$

$\Rightarrow$ $P(x) = -50,000 + 350x$

Now, for $x = 400$

$$P(400) = -50,000 + 350(400) =$$

$$-50,000 + 1,40,000$$

$$= 90,000.$$

3.8 TEST YOUR UNDERSTANDING (B)

1. Let $f = \{(3, 1), (9, 3), (12, 4)\}$ and $g = \{(1, 3), (3, 3), (4, 9), (5, 9)\}$. Show that fog and gof are both defined. Also find fog and gof.

[**Ans.** $gof = \{(3, 3), (9, 3), (12, 9)\}$, $fog = \{(1, 1), (3, 1), (4, 3), (5, 3)\}$]

2. Find $fog(2)$ and $gof(1)$ when $f: R \to R, f(x) = x^2 + 8$ and $g: R \to R, g(x) = 3x^3 + 1$. [**Ans.** $fog(2) = 633$, $gof(1) = 2188$]

3. Let R^+ be the set of all non-negative real numbers. If $R^+ \to R^+$ and $g: R^+ \to R^+$ are define as $f(x) = x^2$ and $g(x) = +\sqrt{x}$. Find fog and gof are they equal functions ?

[**Ans.** $fog(x) = x$, $gof(x) = x$]

4. If $f(x) = x^2 - 2$, $g(x) = 4x + 1$, then describe the following

(i) gof (ii) fog (iii) gog (iv) fof (v) ff (vi) gg

[**Ans.** $gof = 4x^2 - 7$, $fog = 16x^2 + 8x - 1$, $fof = x^4 - 4x^2 + 2$, $gog = 16x + 5$, $ff = x^4 - 4x^2 + 2$, $gg = 16x + 5$]

5. If $f(x) = \frac{x-1}{x+1}$, they verify that $(fof)(x) = -\frac{1}{x}$

6. If $f(x) = 3x - 5$, find $f^{-1}(x)$.
[**Ans.** $f^{-1}(x) = \frac{x+5}{3}$]

7. If a company has a cost function $C(x) = 50 + 20x$ and demand function $p = 75 - 3x$, they find

(i) Revenue function

(ii) Profit function

[Ans. (i) Revenue function $(x) = 75x - 3x^2$
(ii)Pro it function $P(x) = -3x^2 + 55x - 50$]

3.9 LET US SUM UP

- Function is an expression or rule that defines the relationship between two variable quantities out of which one is independent and other one dependent.
- There are various types of functions.
- The algebra of Functions deal with addition and subtraction of functions and the evaluation of the function.
- Functions have many applications in the real business and economics.

3.10 KEY TERMS

- **FUNCTION:**Function is an expression, mapping or rule that defines the relationship between two variable quantities out of which one is independent and other one dependent.
- **DOMAIN AND RANGE:** If f is a mapping from A to B, then A is called the *domain* of the function f and B the *co-domain* of the function f. The image set f(A) is called the Range of the function f.
- **ONE-ONE FUNCTION (INJECTION) :** *A function* f: $A \rightarrow B$*is said to be a one-one function or an injection if different elements of A have different image in B.*

- **MANY-ONE FUNCTION :** *A function* f: $A \rightarrow B$ *is said to be a many-one function of two or more elements of set A have the same image in* B.
- **ONTO FUNCTION (SURJECTION) :** *A function* f: $A \rightarrow B$*is said to be an onto function or a surjection if every element of B is the f-image of some element of* A, i. *e*., *iff* $(A) = B$ *or range of* $\mathbf{f}$ *is the co-domain of* $\mathbf{f}$.
- **INTO FUNCTION :** *A function of* f: $A \rightarrow B$*is an into function if there exists an element in B having no pre-image in*A
- **ONE-ONE INTO MAPPING :** *Any mapping which is one-one as well as into is called one-one into mapping.*
- **SCALAR MATRIX:** A diagonal matrix whose diagonal elements are all equal is called a $scalar\ matrix$.
- **IDENTITY OR UNIT MATRIX (I):** A scalar matrix in which each of its diagonal elements is unity is called an $identity$ or $unit\ matrix$.
- **TRACE OF A MATRIX:** The trace of any square matrix is the sum of its main diagonal elements.
- **TRIANGULAR MATRIX:** If every element above or below the leading diagonal is zero, the matrix is called a $triangular\ matrix$.
- **TRANSPOSE OF A MATRIX**: A matrix obtained by interchanging the corresponding rows and columns of a given matrix.
- **SYMMETRIC MATRIX:** A matrix which is equal to its own transpose.
- **SKEW SYMMETRIC MATRIX:** A matrix which is equal to negative of its own transpose.

3.11 FURTHER READINGS

1. Mizrahi and John Sullivan. *Mathematics for Business and Social Sciences*. Wiley and Sons.
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5. J. K. Singh, *Business Mathematics*, Himalaya Publishing House.

B. COM (DIGITAL)

SEMESTER II

COURSE: BUSINESS MATHEMATICS AND STATISTICS

Unit 4 – Limit and Continuity

STRUCTURE

4.0 OBJECTIVES

After studying the Unit, students will be able to

- Understand the concept of Limit.
- Describe the Rules for finding the limit of a function.
- Apply some standard results of Limits.
- Use various methods to find the Limit of a function.

4 .1 LIMIT – CONCEPT

In case we are find the limit of a function, basically we are concerned with the manner in which a function, say $y = f(x)$, approaches a value as the independent variable 'x' approaches a particular value, say 'a', which may be finite or infinite.

- Understand the Continuity of a Function at a point

When this value 'a' is finite, 'x' can approach 'a' either through value which are lesser than 'a' (to be denoted as $x \to a^-$) or through values which are greater than 'a' (to be denoted as $x \to a^+$).

Thus, we can define the limit of a function as :

Definition. A limit of a function is a value that the function approaches as the independent variable of the function approaches a particular value, i.e. "The limit of the function $f(x)$, as 'x' approaches 'a', is 'l' written as

$$\lim_{x \to a} f(x) = l$$

Also, we can say that

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = l$$

Here, $\lim_{x \to a^-} f(x)$ is termed as Left-Handed Limit (L.H.L.) and $\lim_{x \to a^+} f(x)$ as the Right-Handed Limit (R.H.L.) of the function.

4.2 RULES FOR THE LIMIT OF A FUNCTION

For two functions $f(x)$ and $g(x)$, such that $\lim_{x \to a} f(x) = l_1$ and $\lim_{x \to a} g(x) = l_2$ where l_1 and l_2 are real numbers, we have the following rules :

1. **Sum or Difference Rule**

$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

$$= l_1 \pm l_2$$

(**Note.** If l_1 and l_2 are infinite then $\infty + \infty = \infty$, but $\infty - \infty$ is indeterminate)

2. **Product Rule**

$$\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

$$= l_1 \cdot l_2$$

(**Note.** If l_1 and l_2 are infinite then $\infty \times \infty = \infty$, but one of the l_1 or l_2 is 0, then $0 \times \infty$ is indeterminate)

3. Quotient Rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l_1}{l_2}$$

(l_1 and l_2 are finite and $l_2 \neq 0$) (**Note.** $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are indeterminate)

4. Power Rule

If p and q are integers, then

$$\lim_{x \to a}[f(x)]^{p/q} = [\lim_{x \to a} f(x)]^{p/q}$$

$= l_1^{p/q}$ provided $l^{p/q}$ is a real number.

5. For any real number 'c' i. *e.,* a constant

$$\lim_{x \to a}[c\, f(x)] = c \cdot \lim_{x \to a} f(x) = cl_1$$

6. The limit of a constant function the constant itself

i. *e.,* If $h(x) = c$, a constant then

$$\lim_{x \to a} h(x) = \lim_{x \to a} c = c$$

4.3 SOME IMPORTANT RESULTS ON LIMITS

1. $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$, n being a positive integer.

2. $\lim_{x \to a} (1 + x)^{1/x} = e$

3. $\lim_{x \to a} \left(1 + \frac{1}{x}\right)^x = e$

4. $\lim_{x \to a} \frac{a^x - 1}{x} = \log_e a \ (a > 0)$

5. $\lim_{x \to a} \frac{e^x - 1}{x} = 1$

6. $\lim_{x \to a} \frac{\log(1+x)}{x} = 1$

4.4 METHODS FOR EVALUATION OF LIMITS

4.4.1 Direct Substitution

Example 1. ***Evaluate*** $\lim_{x \to 3} (7x^2 + 5)$

Solution : $\lim_{x \to 3} (7x^2 + 5) = 7(3)^2 + 5 = 68$

Example 2. ***Find the limit of*** $f(x) = 10 - 2x - 5x$, ***when*** x ***approaches*** **2.**

Solution : $\lim_{x \to 2} (10 - 2x - 5x^2) = 10 - 2(2) - 5(2)^2$

$$= 10 - 4 - 20$$

$$= -14$$

4.4.2 Factorisation Method:

It is a technique to find the limit of a function by cancelling out the common factors so as to convert the given function into a determinate form.

Example 3. ***Evaluate*** $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 + 2x - 3}$

Solution : When $x = 1$, we have $\frac{x^2 + x - 2}{x^2 + 2x - 3} = \frac{0}{0}$ form

Now, Factorise the numerator and denominator

$$\Rightarrow \lim_{x \to 1} \frac{(x-1)(x+2)}{(x-1)(x+3)} = \lim_{x \to 1} \frac{(x+2)}{(x+4)} = \frac{3}{4}$$

Example 4. ***Find*** $\lim_{x \to 2} \frac{x^3-8}{x-2}$

Solution : When $x = 2$, we have $\frac{x^3-8}{x-2} = \frac{0}{0}$ form

So, now factorise the numerator and denominator

$$\Rightarrow \lim_{x \to 2} \frac{(x-2)(x^2-2x+4)}{(x-2)} = \lim_{x \to 2} (x^2 - 2x + 4)$$

$$= 2^2 - 2(2) + 4$$

$$= 4$$

4.4.3 RATIONALISATION METHOD: It is a technique by which we find the limit by rationalizing the numerator or denominator that is by multiplying the numerator or denominator by their respective conjugates.

Example 5. ***Evaluate*** $\lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$

Solution : When we put $h = 0$ in the given function, we get $\frac{0}{0}$ form

So, Rationalizing the numerator we get

$$\lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \qquad (\because (a-b)(a+b) = a^2 - b^2)$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$x \to 4 \quad \sqrt{(x^2+9)}-5$

Solution : When $x = 4$, the expression $\frac{x^2-16}{\sqrt{(x^2+9)}-5}$ would be of the form $\frac{0}{0}$.

Example 6. ***Evaluate*** $\lim x2-16$

So, Rationalise the denominator

$$\lim_{x \to 4} \frac{x^2-16}{\sqrt{(x^2+9)}-5} = \lim_{x \to 4} \frac{x^2-16}{\sqrt{(x^2+9)}-5} \times \frac{\sqrt{(x^2+9)}+5}{\sqrt{(x^2+9)}+5}$$

$$= \lim_{x \to 4} \frac{(x^2-16)(\sqrt{x^2+9}+5)}{(x^2+9)-5^2}$$

$$= \lim_{x \to 4} \frac{(x^2-16)(\sqrt{x^2+9}+5)}{(x^2+9-25)}$$

$$= \lim_{x \to 4} \frac{(x^2-16)(\sqrt{x^2+9}+5)}{(x^2-16)}$$

$$= \lim_{x \to 4} (\sqrt{x^2+9}+5)$$

$$= \sqrt{4^2+9}+5 = 10.$$

Example 7. ***Evaluate*** $\lim_{x \to 1} \frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1}$

Solution : When we put $x = 1$ in the given expression, we would obtain $\frac{0}{0}$ form.

So, Ratinalise the numerator

$$\lim_{x \to 1} \frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1} = \lim_{x \to 1} \frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1} \times \frac{\sqrt{3+x}+\sqrt{5-x}}{\sqrt{3+x}+\sqrt{5-x}}$$

$$= \lim_{x \to 1} \frac{(3+x)-(5-x)}{(x^2-1)(\sqrt{3+x}+\sqrt{5-x})}$$

$$(\because (a-b)(a+b) = a^2 - b^2)$$

$$= \lim_{x \to 1} \frac{2x-2}{(x^2-1)(\sqrt{3+x}+\sqrt{5-x})}$$

$$= \lim_{x \to 1} \frac{2(x-1)}{(x-1)(x+1)(\sqrt{3+x}+\sqrt{5-x})}$$

$$= \lim_{x \to 1} \frac{2\times1}{(x+1)(\sqrt{3+x}+\sqrt{5-x})}$$

$$= \frac{2}{(1+1)(\sqrt{3+1}+\sqrt{5-1})}$$

$$= \frac{2}{2(\sqrt{4}+\sqrt{4})} = \frac{1}{(2+2)} = \frac{1}{4}$$

4.4.4 USE OF SOME STANDARD RESULTS

Example 8. ***Evaluate*** $\lim_{x \to 1} \frac{x^{15}-1}{x^{10}-1}$

Solution : When we put $x = 1$ in the expression we would get $\frac{0}{0}$ form.

So, apply the result $\lim_{x \to 1} \frac{x^{15}-1}{x^{10}-1} = na^{n-1}$

For this multiply and divide the given expression by $(x - 1)$

$$\lim_{x \to 1} \frac{\frac{x^{15}-1}{(x-1)}}{\frac{x^{10}-1}{(x-1)}}$$

$$= \frac{15(1)^{15-1}}{10(1)^{10-1}} = \frac{15\times1}{10\times1} = \frac{3}{2}$$

Example 9. ***If*** $\lim_{x \to 2} \frac{x^n - 2^n}{x-2} = 80$ ***When*** $n \in N$***, then find*** 'n'

Solution : $\lim_{x \to 2} \frac{x^n-2^n}{x-2} = 80$

$\Rightarrow$ $n \cdot 2^{n-1} = 80$ $\left(\because \lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right)$

Clearly

$\Rightarrow$ $5 \cdot 2^{5-1} = 80$

$\therefore$ 'n' must be 5 i.e. $n = 5$

Example

10. ***If*** $\lim_{x \to -a} \frac{x^a + a^a}{x+9} = 9$,

then find 'a'.

Solution : $\lim_{x \to -a} \frac{x^a+a^a}{x+9} = 9$

$$\lim_{x \to -a} \frac{x^a-(-a)^9}{x-(-a)} = 9$$

$$9a^{9-1} = 9$$

$\Rightarrow$

$\Rightarrow$ $\left(\because \lim_{x \to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right)$

$\Rightarrow$ $a^8 = 1$

$\Rightarrow$ $a = \pm 1$

Example 11. ***Evaluate*** $\lim_{x \to a} (1 + 5x)^{1/x}$

Solution : $\lim_{x \to a} (1 + 5x)^{1/x}$

$$= \lim_{5x \to a} (1 + 5x)^{1/5x \times 5}$$

$$= \left[\lim_{5x \to a} (1 + 5x)^{1/5x}\right]^5$$

$= e^5 \quad (\because \lim_{x \to 0}(1+x)^{1/x} = e)$

Example 12. ***Evaluate*** $\lim_{x \to 0} \frac{a^x-1}{b^x-1}$

Solution :

$\lim_{x \to 0} \frac{a^x-1}{b^x-1}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x \to 0} \left(\frac{a^x-1}{x} \times \frac{1}{\frac{b^x-1}{x}}\right)$

$= \lim_{x \to 0} \frac{a^x-1}{x} \cdot \frac{1}{\lim_{x\to 0}\frac{b^x-1}{x}}$ $\left(\because \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\right)$

Taking log both sides, we get

$= (\log_e a) \cdot \frac{1}{(\log_e b)}$

$= \frac{\log_e a}{\log_e b} = \frac{\log a}{\log b}$ $\quad$ (using $\lim_{x \to a} \frac{a^x-1}{x} = \log_e a$)

Example 13. ***Evaluate*** $\lim_{x \to 0} \left[\frac{e^x-e^{-x}}{x}\right]$

Solution : $\lim_{x \to 0} \left[\frac{e^x-e^{-x}}{x}\right]$ $\left(\frac{0}{0}\text{ form}\right)$

$\Rightarrow \lim_{x \to 0} \frac{(e^x-1)-(e^{-x}-1)}{x}$

$\Rightarrow \lim_{x \to 0} \left(\frac{e^x-1}{x}\right) + \lim_{x \to 0} \left(\frac{e^{-x}-1}{-x}\right)$

$(\because \lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x))$

$= \log e + \log e$ $\quad (\because \lim_{x \to a} \frac{e^x-1}{x} = \log e = 1)$

$= 1 + 1 = 2$

Example 14. ***Evaluate*** $\lim_{x \to 0} \frac{\log(1+x)}{x}$ **Solution :** $\lim_{x \to 0} \frac{\log(1+x)}{x}$ $\left(\frac{0}{0}\text{form}\right)$

$= \lim_{x \to 0} \frac{1}{x}\log(1+x)$

$= \lim_{x \to 0} \log(1+x)^{1/x}$ $(\because \log m^n = n \log m)$

$= \log\left(\lim_{x \to 0}(1+x)^{1/x}\right)$

$= \log e$ $(\because \lim_{x \to 0}(1+x)^{1/x} = e)$

$= 1$

4.4.5 EVALUATION OF ALGEBRAIC LIMITS WHEN $x \to \infty$

Example 15. $\lim_{x \to \infty} \frac{4x^2+5x+6}{3x^2+4x+5} = ?$

Solution : $\lim_{x \to \infty} \frac{4x^2+5x+6}{3x^2+4x+5}$ $\left(\frac{0}{0}\text{form}\right)$

So, dividing the numerator and denominator by 'x^2', we get

$\lim_{x \to \infty} \frac{4+\frac{5}{x}+\frac{6}{x^2}}{3+\frac{4}{x}+\frac{5}{x^2}}$

$= \frac{4+0+0}{3+0+0}$ $\left(\because \frac{\text{Any number}}{\infty} = 0\right)$

$= \frac{4}{3}$

Example 16. ***Find*** $\lim_{x \to \infty} \frac{(3\ \ -1)(4x-2)}{(x+8)(x-1)}$

Solution : $\lim_{x \to \infty} \frac{(3x-1)(4x-2)}{(x+8)(x-1)}$ $\left(\frac{\infty}{\infty}\text{form}\right)$

So, dividing the numerator and denominator by 'x', we get

$$= \lim_{x \to \infty} \frac{(3-\frac{1}{x})(4-\frac{2}{x})}{(1+\frac{3}{x})(1-\frac{1}{x})}$$

$$= \frac{(3-0)(4-0)}{(1+0)(1-0)}$$

$$= 12$$

$$(\because \frac{\text{Any number}}{\infty} = 0)$$

Example 17. ***Find the limits*** $\lim_{x \to \infty} \frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$

Solution : $\lim_{x \to \infty} \frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$ $\quad (\frac{\infty}{\infty}\text{ form})$

So, dividing each term in the numerator and denominator by 'x', we get

$$= \lim_{x \to \infty} \frac{\sqrt{3-\frac{1}{x^2}}-\sqrt{2-\frac{1}{x^2}}}{4+\frac{3}{x}}$$

$$= \frac{\sqrt{3-0}-\sqrt{2-0}}{4+0}$$

$$= \frac{\sqrt{3}-\sqrt{2}}{4}$$

$$(\because \frac{\text{Any number}}{\infty} = 0)$$

4.5 CONTINUITY AT A POINT

A function $f(x)$ is said to be continuous at a point $x = a$ (of the domain of the function f), if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

and $f(a)$ is finite.

If $f(x)$ is not continuous at a point $x = a$, then it is said to be discontinuous at $x = a$.

Note. A function is discontinuous at $x = a$ in the following cases :

(i) f is not defined at $x = a$,

i. $e.$ (a) does not exist

(ii) $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ does not exist,

$x \to a \quad x \to a$

i.e. either of $L.H.L.$ or $R.H.L.$ does not exist

(iii) Both the limits $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but are not equal,

i.e. $L.H.L. \neq R.H.L.$

(iv) Both the left handed and right handed limits exist, but not equal to $f(a)$. i.e. $\lim_{x \to a} f(x) \neq f(a)$

Example 18. ***Discuss the continuity of the function*** $f(x) = 3x^2 - 2$ ***at*** $x = 2$ **Solution :**

$f(x) = 3x^2 - 2$

$\lim_{x \to 2} f(x)$

$= \lim_{x \to 2} (3x^2 - 2)$

$= 3(2)^2 - 2$

$= 10$

$f(2) = 3(2)^2 - 2 = 10$

$\lim_{x \to 2} f(x) = f(2) = 10$

Now

$\therefore$ Hence f is continuous at $x = 2$.

Example 19. ***Show that*** $f(x) = \begin{cases} 5x - 4, & \text{when } 0 < x \leq 1 \\ 4x^3 - 3x, & \text{when } 1 < x < 2 \end{cases}$

is continuous at $x = 1$.

Solution : We have

$(L.H.L.\text{ at } x = 1) = \lim_{x \to 1^-} f(x)$

$= \lim_{x \to 1}(5x - 4)(\because f(x) = 5x - 4 \text{ when } x \leq 1)$

$= 5 \times 1 - 4 = 1$

$(R.H.L.\text{ at } x = 1) = \lim_{x \to 1^+} f(x)$

$= \lim_{x \to 1}(4x^3 - 3x)(\because f(x) = 4x^3 - 3x \text{ when } x > 1)$

$= 4(1)^3 - 3 \times 1$

$= 4 - 3 = 1$

$f(1) = 5 \times 1 - 4$ $\quad (\because f(x) = 5x - 4 \text{ when } x = 1)$

$= 1$

$\lim_{x \to 1^-} f(x) = f(1) = \lim_{x \to 1^+} f(x)$

And

$\therefore$ So, $f(x)$ is continuous at $x = 1$.

Example 20. ***Discuss the continuity of the function*** $f(x)$ ***at*** $x = 3$**.**

$$f(x) = \begin{cases} 3 - x, & x < 3 \\ 3 + x, & x \geq 3 \end{cases}$$

Solution : We have $(\because f(x) = 3 - x$ when $x < 3)$

$(L.H.L.$ at $x = 3) = \lim_{x \to 3^-} f(x)$

$= \lim_{x \to 3} (3 - x)$ $(\because f(x) = 3 + x$ when $x \geq 3)$

$= 0$

$(R.H.L.$ at $x = 3) = \lim_{x \to 3^+} f(x)$

$= \lim_{x \to 3} (3 + x)$

$= 6$

Clearly $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

Hence, $f(x)$ is discontinuous at $x = 3$.

Example 21. ***For what value of k, the function***

$$f(x) = \begin{cases} \frac{x^4 - 16}{x - 2}, & x \neq 2 \\ k, & x \geq 2 \end{cases}$$

is continuous at $x = 2$

Solution : Since $f(x)$ is continuous at $x = 2$, therefore

$$\lim_{x \to 2} f(x) = f(2)$$

i. *e.* $\lim_{x \to 2} f(x) = k$ …(i)

Now

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^4-16}{x-2}$$

$$= \lim_{x \to 2} \frac{(x^2-4)(x^2+4)}{(x-2)}$$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)(x^2+4)}{(x-2)}$$

$$= \lim_{x \to 2} (x+2)(x^2+4)$$

$$= (2+2)(2^2+4)$$

$$= 4 \times 8$$

$$= 32$$

$\Rightarrow \quad \lim_{x \to 2} f(x) = 32$ …(ii)

By (i) and (ii), we have

$$k = 32$$

4.6 TEST YOUR UNDERSTANDING (A)

---

1. Find the following limits

(a) $\lim_{x \to 2} [x^4 - x^3 + x^2 - 1]$

(b) $\lim_{x \to 1} [1 + x^1 + x^2 + \dots \dots \dots + x^{20}]$

2. Evaluate

(a) $\lim\limits_{x \to 2} \frac{x^2-6x+5}{x^2-3x-10}$

(b) $\lim\limits_{x \to \frac{1}{2}} \frac{(4x^2-1)}{(2x-1)}$

(c) $\lim\limits_{x \to 9} \frac{(x^2-9x)}{(x^2-10x+9)}$

(d) $\lim\limits_{x \to 1} (\sqrt{x+1} - x)$

(e) $\lim\limits_{x \to 0} \frac{x}{\sqrt{a+x}-\sqrt{a-x}}$

3. Evaluate $\lim\limits_{x \to 1} \frac{x^{17}-1}{x^3-1}$

4. Evaluate $\lim\limits_{x \to a} \frac{x\sqrt{x}-a\sqrt{a}}{x-a}$

5. Evaluate the following :

(a) $\lim\limits_{x \to \infty} \frac{ax^2+bx+c}{lx^2+\ \ x+n}$

(b) $\lim\limits_{x \to \infty} \frac{5x^2+2x+1}{7x^2+3x+5}$

(c) $\lim\limits_{x \to \infty} \frac{5x-8}{\sqrt{7x^2+9}}$

6. Find $\lim\limits_{x \to 0} \frac{\sqrt{1+3x}-\sqrt{1-3x}}{x}$

7. Show that $\lim\limits_{x \to 0} (1-4x)^{1/x} = e^{-4}$

8. Evaluate the following limits :

(a) $\lim\limits_{n \to \infty} (1+\frac{1}{n})^{an}$

(b) $\lim\limits_{x \to 0} \frac{5^{6x}-1}{x}$

(c) $\lim\limits_{x \to 0} \frac{e^{-x}-1}{2x}$

9. Discuss the continuity of $f(x)$, where

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$$

at $x = 1$

10. Examine the continuity of the function $f(x)$ at $x = 3$

if $f(x) = \begin{cases} \frac{x-3}{\sqrt{x^2-9}}, & x \neq 3 \\ 0, & x = 3 \end{cases}$

11.

Show that $f(x) = \begin{cases} 1 - x, & x < 0 \\ 1 + x, & x \geq 0 \end{cases}$

is continuous at $x = 0$.

12. For what value of 'k' is the function defined by

$f(x) = \begin{cases} k(x^2 + 2), & \text{if } x \leq 0 \\ 3x + 1, & \text{if } x > 0 \end{cases}$

continuous at $x = 0$?

Answers

1. (a) 11 (b) 21

2. (a) $\frac{4}{7}$ (b) 2 (c) $\frac{9}{8}$ (d) $\frac{1}{\sqrt{2}+1}$ (e) $\sqrt{a}$

$\frac{17}{3}$ (b) $\frac{5}{7}$ (c) $\frac{5}{\sqrt{7}}$

3.

4. $\frac{3}{2}\sqrt{a}$

5. (a) $\frac{a}{l}$

6. 6

8. (a) e^a (b) $6\ \log 5$ (c) $-\frac{1}{2}$

9. Discontinuous at $x = 1$

10. Continuous

12. $k = \frac{1}{2}$

4.7 LET US SUM UP

- The concept of the Limit and Continuity of a function is very important to understand to do calculus.
- Limits basically tells us how does a function behave as we approach a particular value, regardless of the function's actual value there.
- We have different rules and some standard formula which makes it easier to find the Limit of a function as the variable approaches a particular value.
- A function is said to be continuous at a point if both Limit and Function exists at that point and are also equal at that point.

4.8 KEY TERMS

- **LIMIT:** A limit of a function is a value that the function approaches as the independent variable of the function approaches a particular value.
- **CONTINUITY:** A function $f(x)$ is said to be continuous at a point $x = a$ (of the domain of the function f), if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(\quad)$ and $f(a)$ is finite.
- **INDETERMINATE FORMS**: When we cannot find the limit solely from the limits of the individual functions.
- **METHOD OF FACTORIZATION:** It is a technique to find the limit of a function by cancelling out the common factors so as to convert the given function into a determinate form.

- **METHOD OF RATIONALIZATION**: It is a technique by which we find the limit by rationalizing the numerator or denominator that is by multiplying the numerator or denominator by their respective conjugates.

4.9 FURTHER READINGS

1. Mizrahi and John Sullivan. *Mathematics for Business and Social Sciences*. Wiley and Sons.
2. Budnick, P. *Applied Mathematics.* McGraw Hill Publishing Co.
3. N. D. Vohra, *Business Mathematics and Statistics*, McGraw Hill Education (India) Pvt Ltd 4. J.K. Thukral, *Mathematics for Business Studies*, Mayur Publications
5. J. K. Singh, *Business Mathematics*, Himalaya Publishing House.

B. COM (DIGITAL)

SEMESTER II

COURSE: BUSINESS MATHEMATICS AND STATISTICS

Unit 5 – Differentiation

<u>STRUCTURE</u>

5.26 Further Readings

5.0 OBJECTIVES

After studying the Unit, students will be able to

- Understand the concept of Differentiation of a fucntion.
- Find the derivative using definition.
- Differentiate various functions using different formulas.
- Find the second and third order derivatives. • Find the Maxima and Minima of a function.

5.1 INTRODUCTION

If y is a function of only one independent variable, it is called a *univariate function.*

The rate at which a function changes with respect to its independent variable is called the *derivative* in the function. Thus, in more precise words, in case of dependency of y on x (means $y = f(x)$), the differentiation is a method to find the rate at which a quantity 'y' changes with respect to the change in another quantity 'x' upon which it depends. This rate of change is called derivative of y with respect to x.

(a) INCREMENT

In case $y = f(x)$, a small change in the value of variable x is called increment in x and is denoted by Δx or δx (Δx or $\delta x \approx 0$) corresponding to this very small change in x, Δy denotes the small change in variable y.

(b) AVERAGE RATE OF CHANGE

In case $y = f(x)$ the average rate of change of the function f when x changes to $x + \Delta x$ is defined by ratio $\Delta y / \Delta x$.

$\therefore$ Average rate of change of y w.r.t. x

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(c) INSTANTANEOUS RATE OF CHANGE

The limiting value of average rate of change of function $f(x)$ as $\Delta x \to 0$ is known as instantaneous rate of change of function $f(x)$ (i. $e.$, y) w.r.t. x.

5.2 DERIVATIVE OF A FUNCTION y = f(x) w.r.t. x

The instantaneous rate of change of function $f(x) = y$ w.r.t x is called derivative of y w.r.t x i. $e.$, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$ or

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x},$$ provided this limit exists.

Note 1. $\frac{dy}{dx}$ is also called the differential coefficient of y w.r.t x.

Note 2. $\frac{d}{dx}$ represents a single operator and is not to be interpreted as the ratio of two quantities dy and dx.

Note 3. $\frac{dy}{dx}$ is also written as $f'(x)$ or y_1 or Dy $(D = \frac{d}{dx})$

5.3 FINDING DERIVATIVES USING DEFINITION (FIRST PRINCIPLE)

In this section, we shall present some example of finding the derivatives by using the above definition of the derivative. This method of finding the derivative is also called the method of finding the derivative by **first principle** or by **delta method** or by **ab-initio method.**

Example 1. ***Find the derivative of the function*** $y = x^3$ ***from the first principles*** **Solution :** Let $y = f(x) = x^3$. Then, $f(x + h) = (x + h)^3$.

By definition, $$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h},$$ provided the limit exists.

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 + x^3}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3hx + h^2)}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3hx + h^2) = 3x^2$$

Example 2. ***Differentiate*** $f(x) = x^2 + 3x$ ***from definition***

Solution : We have, $f(x) = x^2 + 3x$

$$\therefore \quad f(x+h) = (x+h)^2 + 3(x+h)$$

From first principles, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - \ (x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) - x^2 - 3x}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 + 3x + 3h - x^2 - 3x}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h^2 + 3h}{h} = \lim_{h \to 0} (2x + h + 3) = 2x + 3.$$

Example 3. ***Differentiate from first principles :*** $f(x) = \frac{1}{x^2} (x \neq 0).$

Solution : We are given $f(x) = \frac{1}{x^2}$. Hence, $(x+h) = \frac{1}{(x+h)^2}$.

From first principles, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{x^2 - x^2 - 2hx - h^2}{h}$$

$$= \lim_{h \to 0} \frac{-2hx - h^2}{h(x+h)^2 \cdot x^2} = \lim_{h \to 0} \frac{-2x - h}{(x+h)^2 x^2} = \frac{-2x}{x^4} = -\frac{2}{x^3}.$$

Example 4. ***Differentiate*** $(x) = \sqrt{x}$, $(x > 0)$***, using ab-initio method***

Solution : We are given $f(x) = \sqrt{x}$. Hence, $f(x+h) = \sqrt{(x+h)}$.

Hence, by definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(x+h)}-\sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(x+h)}-\sqrt{x}}{h} \times \frac{\sqrt{(x+h)}+\sqrt{x}}{\sqrt{(x+h)}+\sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h[\sqrt{x+h}+\sqrt{x}]}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h[\sqrt{(x+h)}+\sqrt{x}]} = \lim_{h \to 0} \frac{1}{\sqrt{(x+h)}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

5.4 DERIVATIVES OF SOME STANDARD FUNCTIONS

(i) DIFFERENTIAL COEFFICIENT OF A CONSTANT

Let c be a constant quantity. If $f(x) = c$, we have $f(x + h) = c$, as the constant remains uncharged for every value of x,

$$\therefore \quad f'(x) \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{c-c}{h} = 0.$$

$$\therefore \quad \frac{d}{dx}(c) = 0$$

(ii) DIFFERENTIAL COEFFICIENT OF x^n

Let $f(x) = x^n$; the, $f(x + h) = (x + h)^n$.

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{x^n\left(1+\frac{h}{x}\right)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{x^n}{h}\left[\left(1 + n\cdot\frac{h}{x} + \frac{n(n-1)}{2!}\frac{h^2}{x^2} + \cdots\right) - 1\right]$$

$$\because \quad \frac{d}{dx}x^n = nx^{n-1}$$

(iii) DIFFERENTIAL COEFFICIENT OF THE PRODUCT OF A CONSTANT AND A FUNCTION OF x.

Let c be a constant and $f(x)$ be the given function; then,

$$\frac{d}{dx}[cf(x)] = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h}$$

$$= \lim_{h \to 0} [c.\frac{f(x+h) - f(x)}{h}] = c\frac{d}{dx}f(x)$$

Hence, *the differential coefficient of the product of a constant and function is equal to the product of the constant and the differential coefficient of the function.*

(iv) DIFFERENTIAL COEFFICIENT OF e^x.

Let $f(x) = e^x$; then $f(x+h) = e^{x+h}$.

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{e^{(x+h)} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x}{h}[(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots) - 1] \; [\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots]$$

$$= \lim_{h \to 0} e^x [1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots] = e^x.$$

$$\therefore \quad \frac{d}{dx}e^x = e^x$$

(v) DIFFERENTIAL COEFFICIENT OF log_e x We have

$$\frac{d}{dx}(log_e\, x) = \lim_{h \to 0} \frac{log_e(x+h) - log_e x}{h}$$

$$= \lim_{h \to 0} \frac{log_e(\frac{x+h}{x})}{h} \qquad [\because log_e \; - log_e n = log_e \frac{m}{n}]$$

$$= \lim_{h \to 0} \frac{log_e(1+\frac{h}{x})}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}[\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \cdots] \; [\because log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots]$$

$$= \lim_{h \to 0} [\frac{1}{x} - \frac{h}{2x^2} + \frac{h^2}{3x^3} - \cdots] = \frac{1}{x}.$$

$$\frac{d}{x} log_e x = \frac{1}{x}$$

$\therefore$

(vi) DIFFERENTIAL COEFFICIENT OF log_a x Changing the base from a to e we have

$$\log_a x = \log_e x \,.\, \log_a e.$$

$$\therefore \quad \frac{d}{dx}(\log_a x) = \frac{d}{dx}(\log_e x \,.\, \log_a e)$$

$$= \log_a e \,.\, \frac{d}{dx}(\log_e x) = (\log_a e)\,\frac{1}{x}$$

$$\frac{d}{dx}\log_a x = \frac{\log_a e}{x}$$

$\therefore$

5.5 DIFFERENTIAL COEFFICIENT OF SUM OF TWO FUNCTIONS

The differential coefficient of the sum of the functions is equal to the sum of the differential coefficient of the two functions.

It should be clear that the above method can be extended to the sum or difference of any number of functions. Thus,

$$\frac{d}{dx}[f_1(x) \pm f_2(x) \pm f_3(x) \pm \cdots \pm f_n(x)] = \frac{d}{dx}f_1(x) \pm \frac{d}{dx}f_2(x) \pm \frac{d}{dx}f_3(x) \pm \cdots + - \frac{d}{dx}f_n(x).$$

Example 5. $\frac{d}{dx}(3x^4) = 3 \cdot \frac{d}{dx}x^4 = 3 \cdot 4 \cdot x^{4-1} = 12x^3.$

Example 6. $\frac{d}{dx}(2\sqrt{x}) = 2 \cdot \frac{d}{dx}x^{1/2} = 2 \cdot \frac{1}{2} \cdot x^{\frac{1}{2}-1} = x^{-1/2} = \frac{1}{\sqrt{x}}.$

Example 7. $\frac{d}{dx}(x^3 + 3x^2) = \frac{d}{dx}(x^3) + 3 \cdot \frac{d}{d}(x^2) = 3 \cdot x^2 + 3 \cdot 2x = 3x^2 + 6x.$

Example 8. $\frac{d}{dx}(x^2 + a^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(a^2) = 2x + 0 = 2x.$

Example 9. $\frac{d}{dx}(x^3 + 2\log_e x) = \frac{d}{dx}x^3 + 2\frac{d}{dx}(\log_e x) = 3x^2 + \frac{2}{x}.$

5.6 TEST YOUR UNDERSTANDING (A)

Find out the differential coefficient w.r.t. x of the functions in questions 1 to 7:

2. $x^{1/4}, x^{2/3}, x^{-5/6}$ **1.** $x^2, x^{10}, \frac{1}{x^5}$.

3. **(a)** $(x^3 + 2)$, **(b)** $(2x^3 + 3x^2 + 4)$, **(c)** $(x + \frac{1}{x})$.

4. **(a)** $(x^n + a^n)$, **(b)** $(\sqrt{x} + \sqrt{a})$, **(c)** $a^n(x + b)$, **(d)** $\frac{(x+a)^3}{\sqrt{x}}$.

5. **(a)** $2e^x$, **(b)** $(e^x - 3x^2)$ **(c)** $(4x^3 - 7e^x)$.

6. **(a)** $3\ log_e\ x$ **(b)** $log_e x^2$ **(c)** $log_e x^{-5}$ **(d)** $log_e \sqrt{x}$.

7. $4x^{1/3} + e^x$.

8. If $y = x^3 + 3x^2 + 2x + 1$, Find $\frac{dy}{dx}$ for $x = 0$.

Answers

1. $2x,\ 10x^9,\ -\frac{5}{x^6}$

2. $\frac{1}{4}x^{-3/4}$, $\frac{2}{3}x^{-1/3}$, $\frac{-5}{6}x^{-11/6}$

3. **(a)** $3x^2$ **(b)** $6x(x + 1)$ **(c)** $(1 - x^{-2})$

(d) $(\frac{5}{2}x^{\frac{3}{2}} + \frac{9}{2}ax^{\frac{1}{2}} + \frac{3}{2}a^2x^{-\frac{1}{2}} - \frac{1}{2}a^3x^{-\frac{3}{2}})$

(c) a^n

4. **(a)** nx^{n-1} **(b)** $\frac{1}{2\sqrt{x}}$

5. **(a)** $2e^x$ **(b)** $(e^x - 6x)$ **(c)** $(12x^2 - 7e^x)$

6. **(a)** $\frac{3}{x}$ **(b)** $\frac{2}{x}$ **(c)** $\frac{-5}{x}$ **(d)** $\frac{1}{2x}$

7. $\frac{4}{3}x^{-2/3} + e^x$

8. 2

5.7 DIFFERENTIAL COEFFICIENT OF THE PRODUCT TO TWO FUNCTIONS

If $f_1(x)$ and $f_2(x)$ are the two functions of x, we have

$$\frac{d}{dx}[f_1(\;) \cdot f_2(x)] = f_1(x) \cdot \frac{d}{dx}[f_2(x)] + f_2(x) \cdot \frac{d}{dx}[f_1(x)].$$

The derivative of the product of two functions $=$ *first function* $\times$ *derivative of second function* $+$ *second function* $\times$ *derivative of first function.*

Note :

1. If u and v are two functions of x, the above formula can be written as

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

2. The above formula can be extended to cover the function which is the product of more than two functions. For example, if u, v and w are the functions of x, we have

$$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} - uw\frac{dv}{dx} + vw\frac{du}{dx}$$

5.8 DIFFERENTIAL COEFFICIENT OF THE QUOTIENT OF TWO FUNCTIONS

If $f_1(x)$ and $f_2(x)$ are the functions of x and $f_2(x) \neq 0$, we have

$$\frac{d}{dx}\left[\frac{f_1(x)}{f_2(x)}\right] = \frac{f_2(x)\cdot\frac{d}{dx}[f_1(x)] - f_1(x)\cdot\frac{d}{dx}[f_2(x)]}{[f_2(x)]^2}$$

The derivative of the quotient of two functions is

$$\frac{(Denominator)(Der.\ of\ Numerator) - (Numerator)(Der.\ of\ Denominat\ r)}{(Denominator)^2}$$

Note :

If u and v are two functions of x, then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 10. ***Obtain the derivative of*** $x^2 \cdot log\ x$ ***with respect to*** x**.**

Solution : Let $y = x^2 \cdot \log x$

Then, $\frac{dy}{dx} = x^2 \cdot \frac{d}{dx}(\log x) + (\log x)\frac{d}{dx}(x^2)$

$x^2 \cdot \frac{1}{x} + (\log x)2x = x(1 + 2\log x)$

.

Example 11. ***Find the differential coefficient w.r.t. of*** $(x^2 + 5)(3x + 2)$ ***in two defferent ways, and compare the results.***

Solution : Let

$$y = (x^2 + 5)(3x + 2)$$

$$= 3x^3 + 2x^2 + 15x + 10.$$

Then, $\frac{dy}{dx} = \frac{d}{dx}(3x^3) + \frac{d}{dx}(2x^2) + \frac{d}{dx}(15x) + \frac{d}{dx}(10)$

$$= 9x^2 + 4x + 15 + 0$$

$$= 9x^2 + 4x + 15$$

…(i)

Also, we have

$$y = (x^2 + 5)(3x + 2)$$

$\therefore \quad \frac{dy}{dx} = (x^2 + 5) \cdot \frac{d}{dx}(3x + 2) + (3x + 2) \cdot \frac{d}{dx}(x^2 + 5)$

$$= (x^2 + 5) \cdot 3 + (3x + 2) \cdot 2x$$

$$= 3x^2 + 15 + 6x^2 + 4x$$

$$= 9x^2 + 4x + 15$$

…(ii)

In (i) and (ii) we have obtained of the given function in two different ways. Evidently, both are the same.

Example 12. ***Differentiate*** $\frac{x^2+3}{\sqrt{x}}$ ***with respect to .***

Solution : Let

$$y = x^2 \cdot \log x$$

Then, $$\frac{dy}{dx} = x^2 \cdot \frac{d}{dx}(\log x) + (\log x)\frac{d}{dx}(x^2)$$

$$= x^2 \cdot \frac{1}{x} + (\log x)2x = x(1 + 2\log x)$$

Example 13. ***If*** $= \frac{e^x}{(1+x)}$, ***find*** dy/dx**.**

Solution : $$\frac{dy}{dx} = \frac{(1+x)\cdot\frac{d}{dx}e^x - e^x\cdot\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{(1+x)e^x - e^x(0+1)}{(1+x)^2} = \frac{e^x(1+x-1)}{(1+x)^2} = \frac{x\,e^x}{(1+x)^2}.$$

5.9 TEST YOUR UNDERSTANDING (B)

Find the differential coefficients of the following functions with respect to :

1. $x^3 log_e\, x$, $e^x log_a\, x$, $e^x\sqrt{x}$.

2. $(x^3 + x^4)(e^x log_e\, x)$.

3. $\frac{x}{1+x}$, $\frac{x}{(a^2+x^2)}$

4. $\frac{x^n}{log_e\, x}$, $\frac{e^x}{log_e\, x}$.

5. $\frac{e^x(x-1)}{(x+1)}$, $\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$.

Answers

1. $x^2(1 + 3\,log_e\, x)$, $e^x\left(log_a\, x + \frac{\log_a e}{x}\right)$, $\left[e^x\sqrt{x} + \frac{e^x}{2\sqrt{x}}\right]$

2. $e^x log_e\, x(x^4 + 5x^3 + 3x^2) + e^x(x^3 + x^2)$

3. $\frac{1}{(x+1)^2}$, $\frac{a^2-x^2}{(a^2+x^2)^2}$

4. $\frac{x^{n-1}[n\,log_e x-1]}{(log_e x)^2}$, $\frac{e^x(x\log x-1)}{x(\log x)^2}$

5. $\frac{e^x(x^2+1)}{(x+1)^2}$, $\frac{\sqrt{a}}{\sqrt{x}(\sqrt{a}-\sqrt{x})^2}$.

5.10 DERIVATIVE OF A FUNCTION OF A FUNCTION

Let us first consider what is meant by a function of a function. Let y be a function of t and t be a function of$x$; then, y is called a function of a function of x. Consider the example of the function

$y = (3x^2 + 7x + 8)^{10}$. In this function, let $t = 3x^2 + 7x + 8$. Hence, $y = t^{10}$. Thus, y is a function of t and t is a function of x; hence, y is a function of a function of a x.

Such a function as done not directly depend on the independent variable is called a *function of a function*. We may extend this type of function as a function of a function of a function of …… of a function. For example,

$\{\log\log(ae^x + bx^2)\}^m$ is an extended version of such type of function.

Finding the derivative of a function of a function.

For $v = f(t)$ and $t = \theta(x)$. Now, we are to find $\frac{dy}{dx}$.

Let δt be the change product in t be the change δx in$x$, δy be the change the product in y by the change δt in t and δt be different from zero. Then multiplying and dividing $\frac{\delta y}{\delta x}$ by $\frac{\delta t}{\delta x}$, we get

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta t}\cdot\frac{\delta t}{\delta}$$

When $\delta x \to 0$, each of δt and $\delta y \to 0$ and the functions $\frac{\delta y}{\delta t}$and $\frac{\delta t}{\delta x}$ approach respectively to the

limits $\frac{dy}{dt}$ and $\frac{dt}{dx}$.

Hence, in the limit when $\delta x \to 0$, we have

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\delta y}{\delta t}\times\lim_{\delta x\to 0}\frac{\delta t}{\delta x}$$

Or $$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

This important result is usually referred to as the chain rule or function of a function rule of finding the derivative. This rule is a very important one in finding the derivatives.

If y is a function of u where u is a function of v and v is a function of x, the rule can be extended as under :

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{d} \cdot \frac{dv}{dx}$$

In general, the rule may be expressed as

$$\frac{d\text{ (1st function)}}{dx} = \frac{d\text{ (1st function)}}{d\text{ (2nd function)}} \times \frac{d\text{ (2nd function)}}{d\text{ (3rd function)}} \times \ldots \times \frac{d\text{ (last function)}}{dx}$$

Example 14. ***Find*** $\frac{dy}{dx}$ ***for*** $y = (3x^2 + 7x + 8)^{10}$.

Solution : Let $t = 3x^2 + 7x + 8$. Hence, $y = t^{10}$. Now,

from chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{d(t)^{10}}{dt} \cdot \frac{d}{dx}(3x^2 + 7x + 8)$$

$$= 10t^9 \cdot (6x + 7) = 10(3x^2 + 7x + 8)^9(6x + 7)$$

Note :

In practice, the substitution of t for $3x^2 + 7x + 8$ is done only mentally (and not in writing) and we directly write the answer, as given below:

$$\frac{dy}{dx} = \frac{d}{dx}(3x^2 + 7x + 8)^{10}$$

$$= 10(3x^2 + 7x + 8)^9 \frac{d}{dx}(3x^2 + 7x + 8) = 10(3x^2 + 7x + 8)^9(6x + 7).$$

5.11 DIFFERENTIAL COEFFICIENT OF a^x

We know that $a^x = e^{x \log_e a}$.

$$\therefore \quad \frac{d}{dx}a^x = \frac{d}{dx}e^{x \log_e a}$$

$$= e^{x \log_e a} \times \frac{d}{dx}(x \log_e a)$$

(assuming mentally $t = x \log_e a$ and using chain rule)

$$= e^{x \log_e a} \cdot \log_e a = a^x \cdot \log_e a.$$

$$\frac{d}{dx}a^x = a^x \log_e a$$

∴

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Example 15. ***Differentiate*** $e^{(3x2-6x+2)}$ ***with respect to .***

Solution : Let $y = e^{(3x2-6x+2)}$

Then, we can consider

$$y = e^u, \quad \text{where } u = 3x^2 - 6x + 2.$$

∴ $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 3.2x - 6.1 + 0$

$= 6x - 6.$

By chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= e^u(6x - 6)$$

$$= 6(x - 1)e^{(3x^2-6x+2)}.$$

Example 16. ***Differentiate*** $\log\{x + \sqrt{(x^2 + a^2)}\}$ ***with respect to .***

Solution : Let $y = \log\{x + \sqrt{(x^2 + a^2)}\}$

$$\frac{dy}{dx} = \frac{d}{dx}[\log\{x + \sqrt{(x^2 + a^2)}\}]$$

$$= \frac{1}{\{x+\sqrt{(x^2+a^2)}\}} \cdot \frac{d}{dx}\{x + \sqrt{(x^2 + a^2)}\} \quad \text{(using chain rule)}$$

$$= \frac{1}{\{x+\sqrt{(x^2+a^2)}\}}\left[\frac{d}{dx}(x) + \frac{d}{dx}\{\sqrt{(x^2 + a^2)}\}\right]$$

$$= \frac{1}{\{x+\sqrt{(x^2+a^2)}\}}\left[1 + \frac{1}{2}(x^2 + a^2)^{-1/2} \cdot \frac{d}{dx}(x^2 + a^2)\right]$$

$$= \frac{1}{\{x+\sqrt{(x^2+a^2)}\}}\left[1 + \frac{1}{2}(x^2 + a^2)^{-1/2} \cdot 2x\right]$$

∴

$$= \frac{1}{\{x+\sqrt{(x^2+a^2)}\}}\left[1 + \frac{x}{\sqrt{(x^2+a^2)}}\right]$$

$$= \frac{1}{\{x+\sqrt{(x^2+a^2)}\}}\left[\frac{\{x+\sqrt{(x^2+a^2)}\}}{\sqrt{(x^2+a^2)}}\right]$$

$$= \frac{1}{\sqrt{(x^2+a^2)}}.$$

Example 17. ***Differentiate with respect to*** x ***the function*** $(x^2 + 2)^5(3x^4 - 5)^4$**.**

Solution : Let $y = (x^2 + 2)^5(3x^4 - 5)^4$ ∴

$$\frac{dy}{dx} = (x^2 + 2)^5 \cdot \frac{d}{dx}(3x^4 - 5)^4 + (3x^4 - 5)^4 \cdot \frac{d}{dx}(x^2 + 2)^5$$

(using product rule)

$$= (x^2 + 2)^5 \cdot 4(3x^4 - 5)^3 \cdot \frac{d}{d}(3x^4 - 5) + (3x^4 - 5)^4 \cdot (x^2 + 2)^4 \cdot \frac{d}{dx}(x^2 + 2)$$

(using chain rule)

$$= 4(x^2 + 2)^5(3x^4 - 5)^3 \cdot 12x^3 + (3x^4 - 5)^4 \cdot 5(x^2 + 2)^4 \cdot 2x$$

$$= 2x(x^2 + 2)^4(3x^4 - 5)^3[24x^2(x^2 + 2) + 5(3x^4 - 5)] =$$

$$2x(x^2 + 2)^4(3x^4 - 5)^3(39x^4 + 48x^2 - 25).$$

Example 18. ***Find the derivative of*** $e^{x2/(1+x2)}$ ***with respect to*** x**.**

Solution : Let $y = e^{x2/(1+x2)}$.

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[e^{x^2/(1+x^2)}]$$

$$= e^{x^2/(1+x^2)} \cdot \frac{d}{dx}\left(\frac{x^2}{1+x^2}\right)$$

$$= e^{x^2/(1+x^2)}\left[\frac{(1+x^2)\cdot\frac{d}{dx}x^2 - x^2\cdot\frac{d}{dx}(1+x^2)}{(1+x^2)^2}\right]$$

$$= e^{x^2/(1+x^2)}\left[\frac{(1+x^2)\cdot 2x - x^2\cdot 2x}{(1+x^2)^2}\right]$$

$$= e^{x^2/(1+x^2)}\left[\frac{2x(1+x^2-x^2)}{(1+x^2)^2}\right]$$

$= \frac{2x}{(1+x^2)^2} \cdot e^{x^2/(1+x^2)}$.

Example 19. ***If*** $= \sqrt{\frac{1-x}{1+x}}$***, show that*** $(1 - x^2)\frac{dy}{dx} + y = 0$

Solution : Given $y = \sqrt{\frac{1-x}{1+x}}$, differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2} \cdot \frac{d}{x}\left(\frac{1-x}{1+x}\right)$$

$$= \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2} \cdot \frac{(1+x)(0-1)-(1-x)(0+1)}{(1+x)^2}$$

$$= \frac{1}{2}\frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{-1-x-1+x}{(1+x)^2} = \frac{1}{\sqrt{1-x}(1+x)^{3/2}}$$

$$\therefore\ (1 - x^2)\frac{dy}{dx} + y = (1 - x^2) \cdot \frac{1}{\sqrt{1-x}(1+x)^{3/2}} + \sqrt{\frac{1-x}{1+x}}$$

$$= -\frac{(1-x)(1+x)}{\sqrt{1-x}(1+x)^{3/2}} + \sqrt{\frac{1-x}{1+x}}$$

$$= -\sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1-x}{1+x}} = 0.$$

Example 20. ***Find the derivatives of the following functions :***

(iii) $\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x}$

(i) $log\ (x + \sqrt{x^2 - a^2})$ ***(ii)*** $\left(\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}\right)$

Solution : (i) Let $y = log\ (x + \sqrt{x^2 - a^2})$, differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2-a^2}} \cdot \frac{d}{dx}(x + \sqrt{x^2 - a^2})$

$= \frac{1}{x+\sqrt{x^2-a^2}} \cdot [1 + \frac{1}{2}(x^2 - a^2)^{-1/2} \cdot 2x]$

$= \frac{1}{x+\sqrt{x^2-a^2}} \cdot (1 + \frac{x}{\sqrt{x^2-a^2}})$

$= \frac{1}{x+\sqrt{x^2-a^2}} \cdot \frac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}} = \frac{1}{\sqrt{x^2-a^2}}$.

(ii) Let

$y = \log(\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}) = \log(\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} \times \frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}})$

$= \log(\frac{(x+1)+(x-1)+2\sqrt{x+1}+\sqrt{x-1}}{(x+1)-(x-1)}) = \log(\frac{2x+2\sqrt{x^2-1}}{2})$

$= \log(x + \sqrt{x^2 - 1})$, differentiating w.r.t. x , we get

$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2-1}} \cdot [1 + \frac{1}{2}(x^2 - 1)^{-1/2} \cdot 2x]$

$= \frac{1}{x+\sqrt{x^2-1}} \cdot (1 + \frac{x}{\sqrt{x^2-1}}) = \frac{1}{x+\sqrt{x^2-1}} \cdot \frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}$.

(iii) Let

$y = \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x} = \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x} \times \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}-x}$

$= \frac{(x^2+1)+x^2-2x\sqrt{x^2+1}}{(x^2+1)-x^2} = \frac{2x^2+1-2x\sqrt{x^2+1}}{1}$

$= 2x^2 + 1 - 2x\sqrt{x^2 + 1}$, differentiating w.r.t. x , we get

$\frac{dy}{dx} = 2.2x + 0 - 2\,[x \cdot \frac{1}{2}(x^2 + 1)^{-1/2} \cdot 2x + \sqrt{x^2 + 1} . 1]$

$= 4x - 2\,[\frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2 + 1}]$

$= 4x - 2 \cdot \frac{x^2+(x^2+1)}{\sqrt{x^2+1}} = 2\,[2x - \frac{2x^2+1}{\sqrt{x^2+1}}]$

Example 21. ***If*** $y = (x + \sqrt{x^2 - 1})^m$***, prove that*** $(x^2 - 1)(\frac{dy}{dx})^2 = m^2y^2$**.**

Solution : Given $y = (x + \sqrt{x^2 - 1})^m$ …(i)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = m\cdot\left(x+\sqrt{x^2-1}\right)^{m-1}\cdot\left[1+\frac{1}{2}(x^2-1)^{-1/2}\cdot 2x\right]$$

$$= m\cdot\left(x+\sqrt{x^2-1}\right)^{m-1}\cdot\left(1+\frac{x}{\sqrt{x^2-1}}\right)$$

$$= m\cdot\left(x+\sqrt{x^2-1}\right)^{m-1}\cdot\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}$$

$$= m\cdot\frac{\left(x+\sqrt{x^2-1}\right)^{m}}{\sqrt{x^2-1}} = \frac{my}{\sqrt{x^2-1}} \text{ [using (i)]}$$

On squaring both sides, we get

$$\left(\frac{dy}{dx}\right)^2 = \frac{m^2y^2}{x^2-1} \Rightarrow (x^2-1)\left(\frac{dy}{dx}\right)^2 = m^2y^2.$$

Example 22. ***If*** $y = e^{3\log x+2x}$***, prove that*** $\frac{dy}{dx} = x^2e^{2x}(2x+3)$.

Solution : Given $y = e^{3\log x+2x} = e^{\log x^3}\cdot e^{2x} = x^3\cdot e^{2x}$ $(\because e^{\log x} = x,\ x>0)$ Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = x^3\cdot e^{2x}\cdot 2 + e^{2x}\cdot 3x^2$$

$$= x^2e^{2x}(2x+3).$$

5.12 TEST YOUR UNDERSTANDING (C)

Write down the differential coefficients of the following functions with respect to :

1. $(3-2x)^5$, $(e^x)^3$, $(\log_e x)^3$.

2. e^{x^3}, $\log(x^n+a)$, $\log(e^x+1)$, $\sqrt{\log x}$.

3. **(a)** $\log_e\{x+\sqrt{(x^2+1)}\}$, **(b)** $\log_e(x+\sqrt{x^2+a^2})$.

4. $\log_e\left(x+\frac{1}{x}\right)$.

5. **(a)** $\frac{e^x+e^{-x}}{e^x-e^{-x}}$, **(b)** $\sqrt{\left(\frac{1+x}{1-x}\right)}$.

Answers

1. $-10(3-2x)^4$, $3e^{3x}$, $3(\log x)^2/x$;

2. $e^{x^3} \cdot 3x^2$, $\frac{nx^{n-1}}{(x^n+a)}$, $\frac{e^x}{e^x+1}$, $\frac{1}{2x\sqrt{\log x}}$;

3. **(a)** $\frac{1}{\sqrt{x^2+1}}$ **(b)** $\frac{1}{\sqrt{(x^2+a^2)}}$;

4. $-\frac{(x^2-1)}{x(x^2+1)}$;

5. **(a)** $\frac{-4}{(e^x+e^{-x})^2}$ **(b)** $\frac{1}{(1-x)\sqrt{(1-x^2)}}$.

5.13 FINDING DERIVATIVE USING $\frac{dy}{dx} \times \frac{dx}{dy} = 1$ (WHEN NO DERIVATIVE IS ZERO)

Example 23. ***Find*** $\frac{dy}{dx}$ ***for*** $x = 3y^2 - 4y + 8$.

Solution : We are given $x = 3y^2 - 4y + 8$

Differentiating both sides with respect to y, we get

$$\frac{dx}{dy} = 6y - 4.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{6y-4}.$$

5.14 LOGARITHMIC DIFFERENTIATION

Under the logarithmic differentiation, we first take a log of both sides of the function and then differentiate both the sides with respect to the given variable. Finally the equation is solved for the required derivative. This method is useful in the following cases :

(i) When the function to be differentiate has the variable as in index and

(ii) When the function to be differentiated is a product or quotient of many factors.

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Example 24. ***(When variable occurs as in index) find*** $\frac{dy}{dx}$ ***for the following :***

(i) $y = x^x$ ***(ii)*** $y = x^{xx}$

Solution : (i) We are given $y = x^x$.

Taking the logarithm of both sides, we get

$$\log y = x \log x$$

Differentiating both sides w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{d}{dx}\log x + (\log x) \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \quad \frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x$$

$$\Rightarrow \quad \frac{dy}{dx} = y(1 + \log x) = x^x(1 + \log x).$$

(ii) We are given $y = x^{xx}$.

Taking the logarithm of both sides, we get

$$\log y = x^x \log x$$

Differentiating both sides w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} = x^x \frac{d}{dx}\log x + (\log x) \frac{d}{dx}x^x$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = x^x \frac{1}{x} + (\log x)x^x(1 + \log x)$$

(substituting the value of $\frac{d}{dx}x^x$ obtained in the part (i) of the question)

$$\Rightarrow \quad \frac{dy}{dx} = y[x^{x-1} + x^x(1 + \log x) \log x]$$

$$= x^{xx}[x^{x-1} + x^x(1 + \log x) \log x]$$

Example 25. ***(When the function is a product or quotient of several factors).***

Find $\frac{dy}{dx}$ ***for*** $= \frac{(3x^2+4x)(8x-7)^3}{(2x^3+6)^2(3x^2-10)}$.

Solution : Taking the log of both sides of the given function, we have

$$log_e\, y = log_e(3x^2 + 4x) + 3\, log_e(8x - 7) - 2\, log_e(2x^3 + 6) - log_e(3x^2 - 10).$$

Differentiating both sides w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{6x+4}{3x^2+4x} + \frac{24}{8x-7} - \frac{12x^2}{(2x^3+6)} - \frac{6x}{3x^2-10}$$

$\therefore \quad \frac{dy}{dx} = y\left[\frac{6x+4}{3x^2+4x} + \frac{24}{8x-7} - \frac{12x^2}{(2x^3+6)} - \frac{6x}{3x^2-10}\right].$

Example 26. ***Find*** $\frac{dy}{dx}$ ***, if*** $y = x^{ex}$.

Solution : $y = x^{ex} = (x)^{ex}$

$\therefore \quad \log_e y = e^x \log_e x$

Differentiating both sides w.r.t. x , we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = e^x \cdot \frac{d}{dx}(\log_e x) + \log_e x \cdot \frac{d}{dx}(e^x)$$

$$= e^x \cdot \frac{1}{x} + \log_e x \cdot e^x = e^x \left(\frac{1}{x} + \log_e x\right)$$

$\therefore \quad \frac{dy}{dx} = y \cdot e \quad \left(\frac{1}{x} + \log_e x\right) = x^{ex} \cdot e^x \left(\frac{1}{x} + \log_e x\right)$

Example 27. ***Find the derivative of*** $e^{x2/(1+x2)}$ ***with respect to*** x.

Solution : We have $e^{x2/(1+x2)}$

$\therefore \quad \log_e y = \log_e e^{x^2/(1+x^2)} = \frac{x^2}{1+x^2} \cdot \log_e e = \frac{x^2}{1+x^2} \qquad (\text{since } \log_e e = 1)$

Differentiating both sides with respect to x , we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{(1+x^2)\cdot\frac{d}{dx}x^2-x^2\cdot\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$=\frac{(1+x^2)\cdot 2x-x^2\cdot 2x}{(1+x^2)^2}$$

$$=\frac{2x(1+x^2-x^2)}{(1+x^2)^2}=\frac{2x}{(1+x^2)^2}$$

$$\frac{dy}{dx}=y\cdot\frac{2x}{(1+x^2)^2}=\frac{2x\,e^{x^2/(1+x^2)}}{(1+x^2)^2}$$

∴ .

Example 28. ***Find*** $\frac{dy}{dx}$ ***for the function*** $x^y\cdot y^x=k^2$***, where*** k ***is a constant.***

Solution : We have $x^y\cdot y^x=k^2$

Taking the logarithm of both sides, we get

$$\log x^y+\log y^x=\log k^2$$

or $\quad y\log x+x\log y=2\log k.$

Differentiating both sides with respect to x, we have

$$y\cdot\frac{d}{dx}(\log x)+(\log x)\frac{y}{dx}+x\cdot\frac{d}{dx}(\log y)+(\log y)\frac{d}{dx}(x)=0$$

or $\quad y\cdot\frac{1}{x}+(\log x)\frac{dy}{dx}+x\cdot\frac{1}{y}\cdot\frac{dy}{dx}+\log y=0$

or $\quad\left(\frac{y}{x}+\log y\right)+\left(\log x+\frac{x}{y}\right)\frac{dy}{dx}=0$

∴ $\quad\frac{dy}{dx}=-\frac{(y/x+\log y)}{(\log x+x/y)}.$

Example 29. ***If*** $x^y+y^x=a^b$***, find*** $\frac{dy}{dx}$**.**

Solution : Let $u=x^y$ and $v=y^x$.

Hence, the given equation becomes

$$u+v=a^b.$$

Differentiating with respect to x, we have

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \quad \ldots\text{(i)}$$

Now, $u = x^y$

$\therefore$ $\log u = \log x^y$

$= y \log x.$

Differentiating both sides with respect to x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = y \cdot \frac{d}{dx}(\log x) + (\log x)\frac{dy}{dx}$$

$$= \frac{y}{x} + (\log x)\frac{dy}{dx}$$

$\therefore$.
$$\frac{du}{dx} = u\left[\frac{y}{x} + (\log x)\frac{dy}{dx}\right] = {}^{y}\left[\frac{y}{x} + (\log x)\frac{dy}{dx}\right]$$

Again $v = y^x$

$\therefore$ $\log v = \log y^x$

$= x \log y.$

Differentiating with respect to x,

$$\frac{1}{v} \cdot \frac{dv}{dx} = x \cdot \frac{d}{dx}(\log y) + (\log y)\frac{d}{dx}(x)$$

$$= x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + \log y$$

$\therefore$ $\frac{dv}{dx} = v\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] = y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right].$

Putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (i), we get

$$x^y\left[\frac{y}{x} + (\log x)\frac{dy}{dx}\right] + y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] = 0$$

or $x^{y-1}y + x^y(\log x)\frac{dy}{dx} + xy^{x-1}\frac{dy}{dx} + y^x \log y = 0$

or $[x^y \log x + xy^{x-1}]\frac{dy}{dx} = -[x^{y-1}y + y^x \log y]$

$\therefore$ $\frac{dy}{dx} = -\frac{x^{y-1} + y^x \log y}{x^y \log x + x\, y^{x-1}}.$

5.15 TEST YOUR UNDERSTANDING (D)

In questions 1 & 2, find the differential coefficients with respect to :

1. e^{x^x}, 7^{x^2+2x}.

2. $(\log x)^x$

3. If $y = x^{y^x}$, prove that $\frac{dy}{dx} = \frac{y \log y(1+x \log x \log y)}{x \log x \ (1-\log y)}$.

4. If $y = x^a + a^x + x^x$, find $\frac{dy}{dx}$.

5. Find $\frac{dy}{dx}$ for $x = 3y^3 - 2y^2 + 8$.

Answers

1. $e^{x^x} \cdot x^x(1 + \log_e x)$, $2(\log_e 7)(x + 7) \cdot 7^{x^2+2x}$;

2. $(\log x)^2[\log(\log x) + (1/\log x)]$;

4. $ax^{a-1} + a^x \log_e a + x^x(1 + \log_e x)$;

5. $\frac{1}{9y^2-4y}$.

5.16 DIFFERENTIATION OF IMPLICIT FUNCTIONS

If we are to differentiate an *implicit function* $f(x, y) = 0$, we may sometimes be able to solve it for x or y and proceed explicitly. Quite often it is found inconvenient or even impossible to solve the equation $f(x, y) = 0$ either for x or for y. In such cases, the derivative, viz., $\frac{dy}{dx}$ can be obtained by the following method.

WORKING RULE: Differentiate both sides of the equation $f(x, y) = 0$ with respect to x, remembering that y is the function of x. This will introduce the factor $\frac{dy}{dx}$ or any term involving y. After the differentiation has been completed, solve the resulting equation for $\frac{dy}{dx}$.

5.17 DIFFERENTIATION OF PARAMETRIC EQUATIONS

Sometimes x and y both are expressed in terms of a third variable. The third variable is called a **parameter.** In such questions, $\frac{dy}{dx}$ can be obtained by the use of chain rule.

Thus, if $y = f(t)$ and $x = g(t)$, the derivative of y with respect to x is given by

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{dy}{dt} \Big/ \frac{dx}{dt}, \text{ which is more convenient form.}$$

Example 30. ***Find*** $\frac{dy}{dx}$ ***at*** $x = 1, y = 1,$ ***if*** $x^3 - 2x^2y^2 = 5 - y - 5x$.

Solution : We have

$$x^3 - 2x^2y^2 = 5 - y - 5x$$

or $\quad x^3 - 2x^2y^2 + 5x + y - 5 = 0.$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(x^3) - 2\frac{d}{dx}(x^2y^2) + 5\frac{d}{dx}(x) + \frac{d}{dx}(y) - \frac{d}{dx}(5) = 0$$

or

$$3x^2 - 2\left[x^2\frac{d}{dx}(y^2) + y^2 \cdot \frac{d}{dx}(x^2)\right] + 5 + \frac{dy}{dx} = 0$$

or $\quad 3x^2 - 2\left[x^2 \cdot 2y\frac{dy}{dx} + y^2 \cdot 2x\right] + 5 + \frac{dy}{dx} = 0$

or $\quad (1 - 4x^2y)\frac{dy}{dx} = -(3x^2 - 4xy^2 + 5)$

$$\frac{dy}{dx} = \frac{3x^2 - 4xy^2 + 5}{4x^2y - 1}.$$

∴ (Taking − common)

1

$\therefore$ At $x = 1, y = 1$,

$$\frac{dy}{dx} = \frac{3.1^2 - 4.1.1^2 + 5}{4.1^2.1 - 1} = \frac{4}{3}\cdot$$

Example 31. ***If*** $x^y = e^{x-y}$***, prove that*** $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$ **.**

Solution : We have

$$x^y = e^{x-y}$$

Taking the logarithm of both sides, we get

$$y \log_e x = (x-y) \log_e e$$

$$= x - y \qquad \text{(since } \log_e e = 1\text{)}$$

or $\quad y \log x + y = x$ or $\quad y(1 + \log x) = x$

$$y = \frac{x}{(1+\log x)}$$

or .

Differentiating both sides with respect to x , we get

$$\frac{dy}{dx} = \frac{(1+\log x)\cdot \frac{dy}{dx}(x) - x\cdot \frac{d}{dx}(1+\log x)}{(1+\log x)^2}$$

$$= \frac{(1+\log x)\cdot 1 - x(\frac{1}{x}+0)}{(1+\log x)^2}$$

$$= \frac{(1+\log x)-1}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}\ .$$

Example 32. ***If*** $x^m y^n = (x+y)^{m+n}$***, prove that*** $\frac{dy}{dx} = \frac{y}{x}$**.**

Solution : We have $\quad x^m y^n = (x+y)^{m+n}$

Taking the logarithm of both sides with respect to the base e, we get

$$m \log_e x + n \log_e y = (m+n) \log_e (x+y).$$

Differentiating both sides with respect to x , we get

$$m \cdot \frac{1}{x} + n \cdot \frac{1}{y} \cdot \frac{dy}{dx} = (m+n) \cdot \frac{1}{x+y} \cdot \frac{d}{dx}(x+y)$$

or $\quad \frac{m}{x} + \frac{n}{y} \cdot \frac{dy}{dx} = \frac{(m+n)}{x+y}\left(1 + \frac{dy}{dx}\right)$

or $\left(\frac{n}{y} - \frac{m+n}{x+y}\right)\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}$

or $\left\{\frac{(nx+ny-my-ny)}{y(x+y)}\right\}\frac{dy}{dx} = \frac{(mx+nx-mx-my)}{x(x+y)}$

or $\left(\frac{nx-my}{y}\right)\frac{dy}{dx} = \frac{nx-my}{x}$

$$\frac{dy}{dx} = \frac{y}{x}$$

∴ .

Example 33. ***If*** $x = t \log t$ ***and*** $y = (\log t)/t$***, find*** $\frac{dy}{dx}$ ***for*** $t = 1$.

Solution : We have $x = t \cdot \log t$

∴ $\frac{dx}{dt} = t \cdot \frac{d}{dt}(\log t) + \log t \cdot \frac{d}{dt}(t)$

$$= t \cdot \frac{1}{t} + \log t \cdot 1 = 1 + \log t.$$

Again, $y = \frac{\log t}{t}$

∴ $$\frac{dy}{dt} = \frac{t \cdot \frac{d}{dt}(\log t) - (\log t)\frac{d}{dt}(t)}{t^2}$$

$$= \frac{t \cdot \frac{1}{t} - (\log t)1}{t^2} = \frac{1-\log t}{t^2}$$

Now, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(1-\log t)/t^2}{(1+\log t)} = \frac{1-\log t}{t^2(1+\log t)}$.

At $t = 1$, $\frac{dy}{dx} = \frac{1-\log t}{t^2(1+\log t)} = \frac{1-0}{1+0} = 1$.

Example 34. ***Differentiate*** $x^{\log x}$ ***w.r.t.*** $t = e^x$**.**

Solution : Let $y = x^{\log x}$ and $t = e^x$. Hence, we are to find dy/dt, which is given by

$$\frac{dy}{dt} = \frac{dy}{dx} \div \frac{dt}{dx} \quad \ldots\text{(i)}$$

Now, $y = x^{\log x}$

Taking log on both sides,

$\Rightarrow \quad \log y = (\log x) \log x = (\log x)^2.$

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} \cdot \log x \Rightarrow \frac{dy}{dx} = \frac{2y}{x} \cdot \log x = \frac{2}{x} \cdot x^{\log x . \log x}.$$

Again, $t = e^x \Rightarrow \frac{dt}{dx} = e^x.$

Hence, $\frac{dy}{dt} = \frac{2}{x} \cdot x^{\log x} \cdot \log x \div e^x$ [substituting the values in (i)]

$$= \frac{2x^{\log x} \cdot \log x}{x\, e^x}.$$

5.18 TEST YOUR UNDERSTANDING (E)

Find $\frac{dy}{dx}$ in questions 1 & 2, when

1. $x^3 + 3axy + y^3 = a^3.$

2. (i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$

(ii) $3x^4 - x^2y + 2y^3 = 0.$

3. If $y = x^y$, prove that $\frac{dy}{dx} = \frac{y^2}{x(1 - y \log_e x)}$.

4. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

[**Hint :** Transpose, square and solve for y. Reject the root $y = x$ which does not satisfy the given equation and differentiate the other root to obtain the desired result.]

5. If $x = \frac{3at}{1+t^3}$ and $= \frac{3at^2}{1+t^3}$, show that $\frac{dy}{dx} = \frac{t(2-t^3)}{1-2t^3}$.

6. Differentiate e^x with respect to $\sqrt{x}$.

7. Differentiate $\log_{10} x$ with respect to x^2.

$y = \sqrt{x}^{\sqrt{x}^{\sqrt{x} \ldots to \infty}}$

8. If , prove that $x\frac{dy}{dx} = \frac{y^2}{2(1-y\log_e\sqrt{x})}$.

Answers

1. $-\frac{(x^2+ay)}{(ax+y^2)}$

(i) $-\frac{ax+hy+g}{hx+by+f}$

(ii) $\frac{2x(6x^2-y)}{x^2-6y^2}$

2.,

6. $2\sqrt{x}.\, e^x$

7. $\frac{\log_{10} e}{2x^2}$

5.19 SUCCESSIVE DIFFERENTIATION

As observed in many of the preceding examples, the derivative of a function of x is, in general, also a function of x. This derivative, which may be called the *first derivative* of the function, may itself be differentiated. The result is accordingly called the *second derivative* of the original function. If the second derivative is differentiated, the result is called the *third derivative*, and so on. If the operation of differentiation is performed on a function n times in succession, the final result is called the nth derivative of the function.

Notations. If y denotes the function of x, the successive derivatives of y with respect of x are denoted as under:

The first derivative is denoted by $\frac{dy}{dx}$.

The second derivative is denoted by $\frac{d^2y}{dx^2}$.

$$\therefore \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

Similarly, the third derivative is denoted by $\frac{d^3y}{dx^3}$, and so on.

The letter D is frequently used in place of the symbol $\frac{d}{dx}$. The successive derivatives of y are thus denoted as Dy, D^2y, D^3y,…., D^ny.

Instead of the symbol $\frac{d}{dx}$ or D, the following symbols are also used to denote the successive derivatives :

$$y', y'', y''', \ldots, y^n$$

or $\quad f'(x), f''(x), f'''(x), \ldots, f^n(x)$ or $y_1, y_2, y_3, \ldots, y_n$.

Example 35. ***If*** $y = a\,e^{mx} + b\,e^{-mx}$***, show that*** $\frac{d^2y}{dx^2} = m^2y$.

Solution : We have

$$y = a\,e^{mx} + b\,e^{-mx} \quad \ldots\text{(i)}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = a\,e^{mx} \cdot \frac{d}{dx}(mx) + b\,e^{-mx} \cdot \frac{d}{dx}(-mx)$$

$$= a\,me^{mx} - b\,me^{-mx}$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= a\,m\,e^{mx} \cdot \frac{d}{dx}(mx) - b\,m\,e^{-mx} \cdot \frac{d}{dx}(-mx)$$

$$= a\,m^2e^{mx} + b\,m^2e^{-mx}$$

$$= m^2(a\,e^{mx} + b\,e^{-mx})$$

$$= m^2 y \text{ [from (i)]}$$

Example 36. ***If*** $= x^3 \log \frac{1}{x}$***, prove that*** $x \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3x^2 = 0.$

Solution : We have

$$y = x^3 \log \frac{1}{x} = x^3[\log 1 - \log x] = -x^3 \log x.$$

$$\therefore \quad \frac{dy}{dx} = -\left[x^3 \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x^3)\right]$$

$$= -\left[x^3 \cdot \frac{1}{x} + (\log x)3x^2\right] = -x^2 - 3x^2 \log x.$$

$$\therefore \quad x\frac{dy}{dx} = -x^3 - 3x^3 \log x$$

or $\quad x\frac{dy}{dx} = 3y - x^3$

Differentiating again with respect to x, we have

$$x \cdot \frac{d}{dx}\left(\frac{dy}{x}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x) = 3\frac{dy}{dx} - \frac{d}{dx}(x^3)$$

or $\quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 3\frac{dy}{dx} - 3x^2 \quad$ or

$$x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3x^2 = 0.$$

Example 37. ***If*** $y = \log(x + \sqrt{1 + x^2})$***, show that*** $(l + x^2)y_2 + xy_l = 0$

Solution : We have

$$y = \log(x + \sqrt{1 + x^2})$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x+\sqrt{1+x^2}} \frac{d}{dx}[x + \sqrt{1 + x^2}]$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left[1 + \frac{1}{2}(1 + x^2)^{-1/2}\,2x\right]$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left[1 + \frac{x}{\sqrt{1+x^2}}\right]$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left[\frac{\sqrt{(1+x^2)}+x}{\sqrt{1+x^2}}\right]$$

$$= \frac{1}{\sqrt{1+x^2}}$$

or $\sqrt{(1+x^2)}\frac{dy}{dx} = 1$

$$(1+x^2)\left(\frac{dy}{dx}\right)^2 = 1$$

or .

Differentiating both sides with respect to x , we get

$$(1+x^2)2\frac{dy}{dx}\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2(0+2x) = 0$$

$$2(1+x^2)\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 = 0$$

or

Dividing both sides by $\frac{2dy}{dx}$, we get

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$

i. *e.*, $(1+x^2)y_2 + x\,y_1 = 0.$

Example

38. ***If*** $y = (x + \sqrt{1+x^2})^m$***, prove that*** $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0.$

Solution : We have

$$y = \left(x + \sqrt{1 + x^2}\right)^m$$

$$\frac{dy}{dx} = m\left(x + \sqrt{1 + x^2}\right)^{m-1} \frac{d}{dx}\left(x + \sqrt{1 + x^2}\right)$$

$$= m\left(x + \sqrt{1 + x^2}\right)^{m-1} \left[1 + \frac{1}{2}(1 + x^2)^{-1/2} \cdot 2x\right]$$

$$= m\left(x + \sqrt{1 + x^2}\right)^{m-1} \left[1 + \frac{x}{\sqrt{1+x^2}}\right]$$

$$= m\left(x + \sqrt{1 + x^2}\right)^m \cdot \frac{1}{\sqrt{1+x^2}} = \frac{my}{\sqrt{1+x^2}}$$

$$\sqrt{(1 + x^2)}\frac{dy}{dx} = my$$

$$(1 + x^2)\left(\frac{dy}{dx}\right)^2 = m^2y^2$$

∴

or

or

Differentiating both sides with respect to x.

$$(1 + x^2) \cdot 2\left(\frac{dy}{dx}\right) \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 \cdot 2x = m^2 \cdot 2y\frac{dy}{dx}$$

Dividing both sides by $2\frac{dy}{dx}$, we get

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = m^2y$$

or $$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$$

Example 39. ***If*** $y = (a + b\text{x})e^{c\text{x}}$***, prove that*** $y_2 - 2cv\ \ + c^2y = 0$.

Solution : Given, $y = (a + bx)e^{cx}$ …(i)

Differentiating w.r.t. x, we get

$$y_1 = (a + bx) \cdot e^{cx} \cdot c + e^{cx} \cdot (0 + b.1)$$

$\Rightarrow$ $y_1 = cy + b\, e^{cx}$ …(ii) [using (i)]

Differentiating again w.r.t. x , we get

$$y_2 = cy_1 + be^{cx} \cdot c = cy_1 + c(y_1 - cy) \qquad \text{[using (ii)]}$$

$\Rightarrow$ $y_2 - 2cy_1 + c^2y = 0.$

Example 40. ***If*** $y = \text{x}^\text{x}$***, prove that*** $\frac{d^2y}{d\text{x}^2} - \frac{1}{y}\left(\frac{dy}{d\text{x}}\right)^2 - \frac{y}{\text{x}} = 0$.

Solution : Given $y = x^x$, taking logarithm of both sides, we get

$\log y = x \log x$, differentiating w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$\Rightarrow$ $\frac{dy}{dx} = y(1 + \log x)$ …(i)

Differentiating again w.r.t. x , we get

$$\frac{d^2y}{dx^2} = y \cdot \left(0 + \frac{1}{x}\right) + (1 + \log x) \cdot \frac{dy}{dx}$$

$\Rightarrow$ $$\frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y} \cdot \frac{dy}{dx} \cdot \frac{dy}{d}$$

$$\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0.$$

[using (i)]

$\Rightarrow$

5.20 TEST YOUR UNDERSTANDING (F)

1. Given $y = x\sqrt{x^2 + 9}$, find $\frac{d^2y}{dx^2}$ at $x = 4$ $[\frac{236}{125}]$

2. If $y = \left(x + \sqrt{x^2 - 1}\right)^m$, prove that $\left(x^2 - 1\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$.

3. If $y = \sqrt{x + 1} - \sqrt{x - 1}$, prove that $(x^2 - 1)y_2 + xy_1 = \frac{1}{4}y$.

4. If $x^3 + y^3 = 3axy$, show that $\frac{d^2y}{dx^2} = \frac{2a^3xy}{(ax-y^2)^3}$.

5.21 APPLICATIONS OF DIFFERENTIATION

In economics and commerce, variation of one quantity y with respect to another quantity x usually described in terms of two concepts :

(i) average concept, and

(ii) marginal concept

The average concept expresses the variation of one quantity over a specified range of values of a second quantity. It is usually measured from zero to a certain selected value.The marginal concept concerns with instantaneous rate of change on the dependent variable y for every small variation of x. Therefore, a marginal concept is precise only when variations in x are made smaller and smaller. Therefore, the $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, i. *e.,* the derivative $\frac{dy}{dx}$ is interpreted as the marginal value of x.

5.21.1 AVERAGE COST AND MARGINAL COST

Let the total cost C of the output x is given by

$$c = f(x)$$

then the average cost is defined as the ratio of the total cost of the output (x) and is denoted by AC.

Thus, Average cost (A.C.) $= \frac{\text{Total Cost}}{\text{Output}} = \frac{c}{x} = \frac{f(x)}{x}$

or Total cost $=$ A.C. $\times\, x$

Now, if the output is increased from x to x, and corresponding total cost becomes $c + \Delta c$, the then increase in cost per unit output is given by the ratio $\frac{\Delta c}{\Delta x}$ and the marginal cost is defined as :

$$\text{Marginal Cost (M.C.)} = \lim_{\Delta x \to 0} \left(\frac{\Delta c}{\Delta x}\right) = \frac{dc}{dx}$$

In other words, marginal cost is the first derivative of the total cost c with respect to output x and is the rate of increase in total cost with increase in output.

Example 41.***The total cost** c(x)**, associated with producing and marketing** x **units of an item is given by :***

$$c(\text{x}) = \cdot 005\text{x}^3 - \cdot 02\text{x}^2 + 30\text{x} + 5000$$

Find (i) the average cost function

(ii) the average cost of output of 10 units

(iii) the marginal cost functions (iv) the marginal cost when 3 units are produced.

Solution :(i) The average cost function (or average cost) A.C.

$$\text{A.C.} = \frac{c(x)}{x} - \frac{\cdot 005x^3 - \cdot 02x^2 + 30x + 5000}{x}$$

$$= \cdot 005x^3 - \cdot 02x^2 + 30x + \frac{5000}{x}$$

(ii) Average cost at $x = 10$ becomes

$$= 0 \cdot 005 \times 10^2 - 0 \cdot 02 \times 10 + 30 + \frac{5000}{10}$$

$$= 0 \cdot 5 - 0 \cdot 2 + 30 + 500 = 530 \cdot 3$$

Hence, average cost $= Rs.\, 530 \cdot 3$.

(iii) The marginal cost function (or marginal cost), M.C. is obtained by differentiating c with respect to x .

Thus, M.C. $= \frac{dc}{dx}$

$$= \frac{d}{dx}(\cdot 005x^3 - \cdot 02x^2 + 30x + 5000)$$
$$= \cdot 005(3x^2) - \cdot 02(2x) + 30$$

$$= 0 \cdot 015x^2 - \cdot 04x + 30$$

(iv) The marginal cost when 3 units are produced is M.C. at $x = 3$.

$$= \cdot 015 \times 3^2 - \cdot 04 \times 3 + 30 =$$

$$0 \cdot 135 - \cdot 12 + 30 = 30 \cdot 45$$

Hence, Marginal Cost at $x = 3$ is $Rs. 30 \cdot 45$.

5.21.2 TOTAL REVENUE, AVERAGE REVENUE AND MARGINAL REVENUE

Let p be the price per unit and x is the number of units of an item sold. Then the total revenue is given by

$$R = p \times x$$

Now, Average Revenue (A.R.) is give by

Average Revenue (A.R.) $= \frac{Total\ Revenue}{x} = \frac{R}{x} = \frac{px}{x} = p$ (price)

Further, Marginal Revenue (M.R.) is given by

Marginal Revenue (M.R.)$= \frac{dR}{dx} = p + x\frac{dp}{dx}$

$$= p\left(1 + \frac{x}{p}\frac{dp}{dx}\right)$$

Example 42. ***The demand function of a monopolistic is given by*** $p = 1500 - 2x - x^2$**.** ***Find marginal revenue for any level of output*** x**.** ***Also find M.R. when (i)*** $x = 5$**,** ***(ii)*** $x = 20$**.**

Solution : The demand function is $p = 1500 - 2x - x^2$

$\therefore$ Total Revenue $= R = p \cdot x = (1500 - 2x - x^2)x = 1500x - 2x^2 - x^3$ M.R.

$$= \frac{dR}{dx} = 1500 - 4x - 3x^2$$

(i) M.R. at $x = 5$ is $1500 - 4 \times 5 - 3 \times 5^2 = 1500 - 200 - 75 = 1405$

(ii) M.R. at $x = 20$ is $1500 - 4 \times 20 - 3 \times 20^2 = 1500 - 80 - 1200 = 220$

5.21. 3 Elasticity of Demand and Supply :

ELASTICITY OF DEMAND

There is a relation between Price of the product and demand of the product. When price change, demand also change. This responsiveness of change in demand to change is price is called price elasticity of demand. So, we can define price elasticity of demand as relative change in demand due to relative change in price. Mathematically:

$$Ep = \frac{Percentage\ Change\ \text{in}\ quantity\ demanded}{Percentage\ change\ \text{in}\ Price}.$$

$$= \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}} = \frac{\Delta Q}{Q} \times \frac{P}{\Delta P} = \frac{\Delta Q}{\Delta P} \times \frac{P}{Q}$$

PRICE ELASTICITY OF SUPPLY:

There is a relation between Price of the product and Supply of the product. When price change, supply also change. This responsiveness of change in supply to change is price is called price elasticity of supply. So, we can define price elasticity of supply as relative change in supply due to relative change in price. Mathematically:

$$Ep = \frac{Percentage\ Change\ \text{in}\ quantity\ supplied}{Percentage\ change\ \text{in}\ Price}.$$

$$= \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}} = \frac{\Delta Q}{Q} \times \frac{P}{\Delta P} = \frac{\Delta Q}{\Delta P} \times \frac{P}{Q}$$

Example43. If the demand function is $q = 100 - 2p - 2p^2$ calculate the slope of demand curve and also find Elasticity of demand when P = 100.

Solution

The demand function is $q = 100 - 2p - 2p^2$

Differentiating wrt p

$\frac{dq}{dp} = \frac{d}{dp}$ (100 – 2p – 2p^2) = – 2 –4p

Slope of the demand cureve = $\frac{dq}{dp}$ =- (– 2 – 4p) =2+4p

(ii) Elasticity of demand = $\frac{p}{p} \times \frac{dq}{dp}$

$= -\frac{p}{100-2p-2p^2} \times (-2 - 4p) = \frac{p(2+4p)}{100-2p-2p^2}$

Elasticity of demand when p = 10

$$-\frac{10(2+40)}{100-20-200} = \frac{-420}{120} = \frac{-70}{2}$$

Example44. For the supply function x = 2 + 3p^2, find the elasticity of supply at p = 3

Solution

The supply function

x = 2 + 3p^2

Thus $\frac{dx}{dp}$ =6p

Now elasticity of supply = $\frac{p}{x}\frac{dx}{dp} = \frac{p}{2+3p^2} \times 6^p = \frac{6p^2}{2+3p^2}$

Elasticity of supply when p = 3

$= \frac{6 \times 9}{2+27} = \frac{54}{29}$

5.22 MAXIMA AND MINIMA

A function $f(x)$ is said to have attained its maximum value at $x = c$, if the function ceases to increase and begins of decrease at $x = c$.

Similarly, a function $f(x)$ is said to have attained its minimum value at $x = d$, if the function ceases to decrease and begins to increase at $x = d$.

Let us suppose that a function $f(x)$ is defined in an interval by such an expression that it is difficult to get an idea about its behavior just by a glance. In such cases, the best procedure is to draw the curve $y = f(x)$. We know that for drawing a curve just plot some individual points and then join these points by a smooth and regular curve.

Suppose that Fig. 1 shows the graph of some function of x on interval $[a, b]$. The points A_2 and A_4 are called *maximum points* of the graph while the points A_1 and A_3 are called *minimum points*. The function has a maximum value A_2 when $x = M_2$, and a maximum value A_4 when $x = M_4$. Again the function has a minimum value A_1 when $x = M_1$ and a minimum value A_3 when $x = M_3$. Notice that in this case the minimum value A_1 is greater then the maximum value A_4.

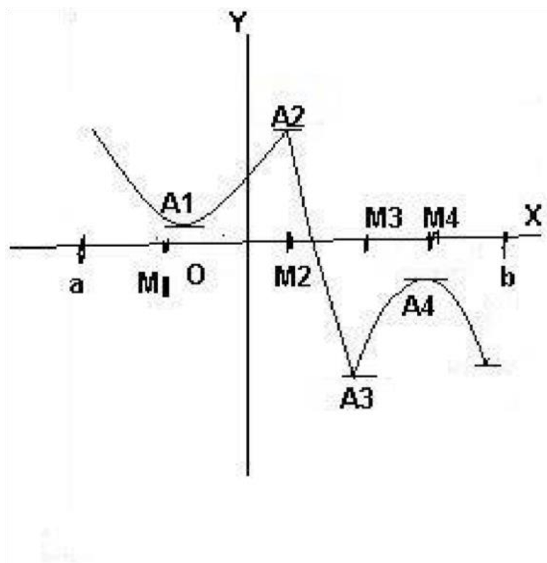

Fig. 1

The following points regarding minima and maxima should be noted carefully :

(i) The maximum and minimum values of a function at a point does not mean the greatest and least values of the function; they only mean the greatest and the smallest value of the function in the immediate neighborhood of that point.

(ii) The function may have several maximum and minimum values, and these must occur alternately.

(iii) Some of the maximum values of the function may be greater than some of its maximum values.

(iv) The maximum and minimum values of the function together are also called the extreme values (extreme) or optimum values (optima) of the function. Minimization and maximization together may be called optimization.

(v) The whole concept of minimum and maximum values discussed above relates to $local$ or $relative$ minimum and maximum values, i. $e.,$ minimum or maximum value in its immediate neighborhood. The $global$ or $absolute$ minimum or maximum value of the function may not be any of them; it (global extreme value) may instead lie the end-point of the interval, as is the case in Fig. 1 in which the overall maximum value lies at $x = a$ and minimum value at $x = b$.

(vi) The terms maxima, minima, extreme and optima are the plural forms of maximum, minimum, extremum and optimum respectively.

CRITERIA FOR MINIMUM AND MAXIMUM

(a) Suppose $f(x)$ has a maximum at $x = c$. Then, by definition, it is an increasing function for the values of x immediately before $x = c$ and a decreasing function for the values of x immediately after $x = c$. If the function and its derivatives are continuous, $f'(x)$ will be positive just before $x = c$ and increasing function after $x = c$. The derivative $f'(x)$ will be negative just before $x = c$, positive just after $x = c$ zero at $x = c$. Thus, the following criteria for the minimum and maximum emerge :

(i) $f(x)$ has a minimum or maximum at a point where $f'(x) = 0$. Let this point be c.

(ii) If $f'(x) > 0$ for x a little less then c and $f'(x) < 0$ for x a little greater then c, $f(x)$ has a maximum value at $x = c$. If $f'(x) < 0$ for x a little less then c and $f'(x) > 0$ for x a little greater then c , $f(x)$ has a minimum value at $x = c$.

(b) If the function $f(x)$ is twice differentiable, the sign of the second derivative at a point c for which $f'(x) = 0$ may be used to determine the minimum and maximum. If $f''(c) > 0$, $f'(c)$ will be changing sign from negative to positive near the point $x = c$ as we move from left to right, i. $e.,$ at $x = c$, $f(x)$ will have a minimum value. If $f''(c) < 0$, $f'(c)$ will be changing sign from positive to negative near the point $x = c$, i. $e.,$ at $x = c$, $f(x)$ will have a maximum value.

Thus, an alternative criteria for the minimum and maximum may be given a under :

(i) $f(x)$ has a minimum or maximum at a point where $f'(x) = 0$. Let this point be c.

(ii) If $f'(c) > 0$, $f(x)$ has a minimum.

If $f'(c) > 0$, $f(x)$ has a maximum.

Notes :

(i) The conditions (i) and (ii) [given under (a) and (b) above] are called **first order condition** and **second order condition** respectively.

(ii) The points at which the first derivative is zero are called **stationary points.** At a stationary point, we have a minimum, maximum or point of inflexion.

Working rule for finding maximum and minimum values of a function :

(I) Find $\frac{dy}{dx}$ for the given function $y = f(x)$.

(II) Find the values of x for which $\frac{dy}{dx}$ is zero. Let these be a_1, a_2, a_3,….

Method 1.

(III) Take the first value a_1. If $\frac{dy}{dx}$ is positive for $x < a_1$ and negative for $x > a_1$, the function has a maximum at $x = a_1$. If $\frac{dy}{dx}$ is negative for $x < a_1$ and positive for $x > a_1$, the function has a minimum at $x = a_1$. Similarly test the other values a_2, a_3, ... of x found in step II.

Method 2.

$\frac{dy}{dx^2}$ 2 (IV)

Find

$x = a_1$ in $\frac{d^2y}{dx^2}$.

(V) Put If the result is negative, the function has a maximum value at

$x = a_1$ in $\frac{d^2y}{dx^2}$,

$x = a_1$ and the maximum value is $y = f(a_1)$. If, by putting the result is positive, the function has a minimum value at $x = a_1$ and the minimum value is $y = f(a_1)$. Similarly test the other values $a_2, a_3, ...$ of x found in step II.

ILLUSTRATIVE EXAMPLES

Example 45. ***Find the maximum and minimum values of the polynomial***

$$8x^5 - 15x^4 + 10x^2$$

Solution : Let $y = 8x^5 - 15x^4 + 10x^2$

$\therefore \quad \frac{dy}{dx} = 40x^4 - 60x^3 + 20x$

For maxima or minima,

$$\frac{dy}{dx} = 0$$

i. e., $40x^4 - 60x^3 + 20x = 0$or .

$20(x-1)^2(2x+1) = 0$

Hence, for y to be maximum or minimum, $x = 0, 1, -1/2$.

Now, $\frac{d^2y}{dx^2} = 160x^3 - 180x^2 + 20$

$= 20(8x^3 - 9x^2 + 1)$

When $= -\frac{1}{2}$, $\frac{d^2y}{dx^2} = -45$ which is negative. Therefore, for this value of x , y has a maximum and that value is $\frac{21}{10}$.

Again, when $x = 0$, $\frac{d^2y}{dx^2} = 20$ which is positive. Therefore, at $x = 0$, y has a minimum which is zero.

Now, at $x = 1$, $\frac{d^2y}{dx^2} = 0$. Hence, for further investigation, we will have to find the next higher derivative.

Now, $\frac{d^2y}{dx^2} = 480x^2 - 360x$.

$\therefore$ at $x = 1$, $\frac{d^2y}{dx^2} = 120$ which is not zero. Hence, at $x = 1$ the function has neither a maximum nor a minimum but a point of inflexion.

Example 46. ***Test the following function for minimum, maximum and point of inflexion :***

$$y = x^3 - 3x + 8$$

Solution : At minimum or maximum,

$$y_1 = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x \pm 1.$$

Now, $y_2 = 6x$.

At $x = 1$, $y_2 = 6 > 0$. Hence, y has a minimum at $x = 1$.

At $x = -1$, $y_2 = -6 < 0$. Hence, y has a maximum at $x = -1$.

At a point of inflexion, $y_2 = 0$ and $y_3 \neq 0$.

Now, $y_2 = 0 \Rightarrow 6x = 0 \Rightarrow x = 0$.

$y_3 = 6 \neq 0$. Hence, y has a point of inflexion at $x = 0$.

Example 47. ***Show that the demand function*** $x = 4a^2 - 3ap^2 + p^3$ ***(where*** p ***is price,*** x ***is quantity,*** a ***is positive constant and*** $p < 2a$***) is downward sloping with a point of inflexion.***

Solution : $\frac{dx}{dp} = -6ap + 3p^2 = -3p(2a - p) < 0$ $(\because 2a > p$ and $p > 0)$

Hence, the function is downward sloping

At the point of inflexion, we have

$$\frac{d^2x}{dp^2} = 0 \quad \text{and} \quad \frac{d^3x}{dp^3} \neq 0.$$

Now $\frac{d^2x}{dp^2} = 0 \Rightarrow -6a + 6p = 0 \Rightarrow p = a$

$$\frac{d^3x}{dp^3} = \frac{d}{dp}[-6a + 6p] = 6 \neq 0.$$

$\therefore$ At $p = a$, the function has a point of inflexion. Thus, the function is downward sloping with a point of inflexion.

Example 48. ***The production manager of a company plans to include*** 180 ***square centimeters of actual printed matter in each page of a book under production. Each page should have*** 2.0 ***cm wide margin along the top and bottom and*** 2.5 ***cm wide margin along the sides. What are the most economical dimensions of each printed page*** **?**

Solution : Let x, y denote the length and breadth of the printed matter in each page. Then, printed area of each page,

$$xy = 180 \qquad \ldots\text{(i)}$$

Due to margin, the dimensions of each page will be $(x + 2 \times 2)$ and $(y + 2 \times 2.5)$, i. *e.*, $(x + 4)$ and $(y + 5)$.

Let A be the total area of each page; then,

$$A = (x+4)(y+5) = xy + 5x + 4y + 20$$
$$= 200 + 5x + 4 \times \frac{180}{x} \qquad \ldots\text{(ii)}$$

For maximum and minimum,

$$\frac{dA}{dx} = 0 \Rightarrow 5 - \frac{720}{x^2} = 0$$

$\Rightarrow \quad x^2 = 144 \Rightarrow x = \pm 12$

$\therefore x = 12$, discarding the negative value.

Again, $\quad \frac{d^2A}{dx^2} = \frac{2 \times 720}{x^2} > 0, \quad$ when $x = 12$

$\therefore x = 12$ minimizes A.

Substituting $x = 12$ in (i), we have

$$y = \frac{180}{12} = 15.$$

Hence, the most economical dimensions are :

length $= x + 4 = 16$ cm and breadth $= y + 5 = 20$ cm.

Example 49. ***The demand function for a particulars commodity is*** $p = 15e^{-x/3}$, $0 \leq x \leq 8$***, where*** p ***is price per unit and*** x ***is the number of units demanded. Determine the price and quantity for which the revenue is maximum.***

Solution : The demand function is $p = 15e^{-x/3}, 0 \leq x \leq 8$

$\therefore$ Total Revenue $= R = p.\ x = 15e^{-x/3}$ $\therefore$

$$\frac{dR}{dx} = 15\,[xe^{-x/3}.(-\tfrac{1}{3}) + 1.\,e^{-x/3}]$$
$$= 15e^{-x/3}\,[-\tfrac{x}{3} + 1] = 15e^{-x/3}\,[-\tfrac{x+3}{3}]$$
$$= e^{-x/3}[15 - 5x]$$
$$\frac{d^2R}{dx^2} = e^{-x/3}(-5) + e^{-x/3}\,(-\tfrac{1}{3})\,(15 - 5x)$$
$$= e^{-x/3}\,[-5 - 5 + \tfrac{5}{3}x] = e^{-x/3}\,[\tfrac{5x}{3} - 10]$$
$$\frac{dR}{dx} = 0 \Rightarrow e^{-x/3}(15 - 5x) = 0 \Rightarrow 15 - 5x = 0 \Rightarrow x = 3$$

and

Now,

Also, $\left.\frac{d^2R}{dx^2}\right]_{x=3}$ gives $e^{-3/3}\left[\frac{5\times3}{3} - 10\right] = -\frac{5}{e} < 0$

$\therefore R$ is maximum when $x = 3$. when $x = 3$,

we have $p = 15e^{-x/3} = \frac{15}{e}$.

Example 50. ***The cost of manufacturing a certain article is given by*** $c = q^2 - 4q + 100$***, where*** q ***is the number of articles manufactured. Find the minimum value of*** c. **Solution :** Here $c = q^2 - 4q + 100$

$\therefore \quad \frac{dc}{dq} = 2q - 4$

Now, for the minimum $\quad \frac{dc}{dq} = 0 \Rightarrow 2q - 4 = 0 \Rightarrow q = 2.$

Also, $\quad \frac{d^2c}{dq^2} = 2 > 0$

Since, $\quad \frac{d^2c}{dq^2} > 0$, therefore, $q = 2$ gives minimum costs.

5.23 TEST YOUR UNDERSTANDING (G)

1. Find the maximum and minimum value of the function $y = \frac{1}{3}x^3 - 2x^2 + 3x + 1$

2. Find the maximum and minimum values of $y = (x - 1)(x - 2)(x - 3)$.

3. The demand equation for a manufacturer's product is $= \frac{80-x}{4}$, where x is the number of units and p is the price per unit. At what value of x will there be maximum revenue ? What is the maximum revenue ?

4. The cost of manufacturing a particular type of Cricket ball is given by

$$c(x) = x^2 - 1200x + 360040$$

Where x denotes the number of balls produced. How many balls should the company manufacture at which cost is minimum, and what would be cost per ball at this level of production ?

5. If the cost function is given by $c = a + bx + cx^2$, where x is the quantity of output. Show that

$$\frac{d}{dx}(\text{A.C.}) = \frac{1}{x}(\text{M.C.} - \text{A.C.})$$

where M.C. and A.C. are marginal cost and average cost respectively.

6. The demand curve for a monopolist is given by $q = 100 - 4p$.

(i) find total average and marginal revenue.

(ii) at what value of q , M.R.= 0

(iii) what is the price when M.R. = 0 ?

7. The demand for a certain product is represented by the equation :

$$p = 20 + 5x - 3x^2$$

where x is the number of units demanded and p is the price per unit.

(i) Find the marginal revenue. (ii) Obtain the marginal revenue when 2 units are sold.

Answers

;

Max $(y)_{x=2-1/3} = \frac{2}{9}\sqrt{3}$, Min. $(y)_{x=2+1/\sqrt{3}} = -\frac{2}{9}\sqrt{3}$

1. Max $(y)_{x=1} = \frac{7}{3}$

2. ;

3. $x = 40, Rs. 400$

4. $x = 600, Rs. 40$

6. (i) $R = 25q - \frac{1}{4}q^2$, A.R. $= 25 - \frac{1}{4}q$, M.R.$= 25 - \frac{1}{2}q$

(ii) M.R. = 0 when $q = 50$

(iii) M.R. $= 0, p = 12 \cdot 5$

7. (i) M.R. $= 20 + 10x - 9x^2$ (ii)
M.R. at $x = 2,4$.

5.24 LET US SUM UP

- In this unit we have discussed the meaning of Derivative of a function as the rate at which a function changes with respect to its independent variable.
- We learnt the method of finding the derivative of a function by using definition (First Principle).
- After that derivative of some standard functions were discussed along with derivative of sum, product and quotient of two functions,
- Derivative of composite, logarithmic, exponential and implicit functions were also discussed.
- Differentiation of second and third order is covered in the unit.
- Economic applications of Revenue, Cost and Elasticity is discussed in the unit.
- Concept of Maxima and Minima is also discussed in the unit.

5.25 KEY TERMS

- **DERIVATIVE:**The rate at which a function changes with respect to its independent variable is called the *derivative* in the function.
- **ELASTICITY OF DEMAND:** The responsiveness of change in demand to change is price is called price elasticity of demand. So, we can define price elasticity of demand as relative change in demand due to relative change in price.
- **PRICE ELASTICITY OF SUPPLY:**The responsiveness of change in supply to change is price is called price elasticity of supply. So, we can define price elasticity of supply as relative change in supply due to relative change in price..
- **MAXIMA:**A function $f(x)$ is said to have attained its maximum value at $x = c$, if the function ceases to increase and begins of decrease at $x = c$.

- **MINIMA**:Similarly, a function $f(x)$ is said to have attained its minimum value at $x = d$, if the function ceases to decrease and begins to increase at $x = d$.

5.26 FURTHER READINGS

1. Mizrahi and John Sullivan. *Mathematics for Business and Social Sciences*. Wiley and Sons.
2. Budnick, P. *Applied Mathematics.* McGraw Hill Publishing Co.
3. N. D. Vohra, *Business Mathematics and Statistics*, McGraw Hill Education (India) Pvt Ltd 4. J.K. Thukral, *Mathematics for Business Studies,* Mayur Publications
5. J. K. Singh, *Business Mathematics*, Himalaya Publishing House.

B. COM (DIGITAL)

SEMESTER II

COURSE: BUSINESS MATHEMATICS AND STATISTICS

Unit 6 – Basic Mathematics of Finance

STRUCTURE

6.0 OBJECTIVES

After studying the Unit, students will be able to

- Describe the concept of Interest.
- Understand the meaning of simple and compound interest.
- Calculate Nominal and Effective rate of interest.
- Apply the concept of compounding and discount in real life situation.

- Apply the concept of continuous compounding.

6.1 INTRODUCTION

In this chapter, our main focus is on the concept of time value of money. Time value of money basically means that the value of a unity of money is different in different time periods. It is based on the concept that the present worth of money received after some time will be less than a money received today. So as the value of money keeps on changing with time, therefore sometimes it becomes necessary to evaluate the value of money at a particular point of time. In the present chapter we will cover such issues.

6.2 INTEREST

Interest is the additional amount paid by a borrower for the use of a lender's money.If you borrow (or lend) some money from (or to) a person for a particular period, you would have to pay (or would receive) more money than your initial borrowing (or lending). This excess or additional money you paid (or received) is called interest.

Interest can be calculated in two ways:

(i) Simple Interest

(ii) Compound Interest

6.3 SIMPLE INTEREST

Simple interest is the interest computed only on the original amount borrowed (or lent). It is the extra money paid (or taken) on that principal for one time period i. *e.,* when the principal remains the same for entire loan period, irrespective of the length of period for which it is borrowed (or lent) then interest is called Simple interest. It is calculated by using following formula :

$$S.I. = \frac{P \times r \times t}{100}$$

Where P = Principal (i. $e.$, the initial amount of borrowing or lending).

r = Rate of Interest (i. $e.$, the interest charged generally for 1 $year$ per Rs. 100).

t= Time period for which money is borrowed or lent(in years).

Note :

For any transaction, time may be given in months, weeks or days. However, in case of $S.I.$ formula, $t$must be in years only. For that the following conversions may be used :

n months $= \frac{n}{12}$years ($\because$ 1 year = 12 months)

m weeks $= \frac{m}{52}$years ($\because$ 1 year = 52 weeks)

$= \frac{k}{365}$

k days years ($\because$ 1 year = 365 days)

Also, A = The total money paid back at the end of specified period.

$\therefore \quad Amount = Principal + Interest$

i. $e.$, $\quad A = P + S.$

$I.$

Example 1. ***Find the simple interest on Rs.*** 2000 ***at*** $6\%\ p.a.$**,** $t = 2\ y\ ars$**.**

Solution : Here, $P =$ Rs. 2000, $r = 6\%\ p.a.$, $t = 2\ years$

$\therefore \quad S.I. = \frac{2000 \times 6 \times 2}{100} = Rs.\,240$.

Example 2. ***Find the simple interest on*** $Rs.\ 10,\ 000$ ***for*** $7\ years$ ***at*** $5\%\ p.\ a.$ ***Find the amount at the end of*** $7\ years$**.**

Solution : Here, $P = 10{,}000$, $r = 5\%\ p.a.$, $t = 7\ years$

$\therefore \quad S.I. = \frac{P \times r \times t}{100} = \frac{10{,}000 \times 5 \times 7}{100}$

$= Rs.\,3500$

$\therefore \quad Amount = P + S.I.$

$= 10{,}000 + 3500$

$= Rs.\ 13{,}500.$

Example 3. ***Find the time required to earn*** $Rs.\ 72,\ 675$ ***as simple interest on the principal of*** $Rs.\ 5,000$ ***at the rate*** $4.5\%\ p.a.$

Solution : Here, $P = 85{,}000, r = 4.5\%\ p.a., t = ?, S.I. = 72{,}675$

We know that, $S.I. = \frac{P\times r\times t}{100}$

$\Rightarrow \quad 72{,}675 = \frac{85{,}000\times 4.5\times t}{100}$

$\Rightarrow \quad T = \frac{72{,}675\times 100}{85{,}000}4.5$

$t = \frac{85.5}{4.5} = 19$

$\Rightarrow$

$\therefore$ The required time is $19\ years.$

Example 4. ***At what rate of interest would the sum*** $Rs.\ 46,\ 875$ ***grow to*** $Rs.\ 50,\ 000$ ***in*** 1 $year$ ***and*** $months$**?**

Solution : Here, $P = Rs.\ 46{,}875$

$A = Rs.\ 50{,}000$

$t = 1\ year$ and $8\ months$

$= 1\frac{8}{12}\ years$

$= 1\frac{2}{3}\ years$

$= \frac{5}{3}\ y\ \ ars$

Now, $Amount = Principal + Simple\ Interest$

i. $e.$ $\quad A = P + S.I.$

$\Rightarrow \quad A = P + \frac{P \times r \times t}{100}$

$\Rightarrow \quad A = P\left(1 + \frac{rt}{100}\right)$

$50{,}000 = 46{,}875\left(1 + \frac{5}{3}i\right)$

$\Rightarrow \left(where\ i = \frac{r}{100}\right)$

$\Rightarrow \quad \frac{50{,}000}{46{,}875} = 1 + \frac{5}{3}i$

$(1.067 - 1) \times \frac{3}{5} = i$

$\Rightarrow$

$\Rightarrow \quad i = 0.04$

$\Rightarrow \quad r = 4\%\ p.a.$

Example 5. ***Ram deposited some amount in a bank at simple interest, such that it would amount to*** $Rs.\ 7650$ ***in*** $4\ years$ ***and*** $Rs.\ 8100$ ***in*** $6\ years$***. Find the sum deposited and the rate of simple interest*** **?**

Solution : Here,

$Principal + S.I.$ for $4\ years = Rs.\ 7650$ …(i)

$Principal + S.I.$ for $6\ years = Rs.\ 8100$ …(ii)

Subtracting (ii) from (i), we obtain

$Simple\ Interest$ for $2\ years = Rs.\ 450$

$\therefore \quad S.I.$ for $1\ year = \frac{450}{2} = Rs.\ 225$

Now, $\quad S.I.$ for $4\ years = Rs.\ (4 \times 225)$

$= Rs.\ 900$ …(iii)

Putting (iii) in (i), we get

$Principal + Rs.\ 900 = Rs.\ 7650$

$\Rightarrow \quad Principal = Rs.\ (7650 - 900) = Rs.\ 6750$

Now, as $S.I. = \frac{P \times r \times t}{100}$

$\therefore \quad r = \frac{S.I. \times 100}{P \times t}$

For 4 $years$, $S.I. = s.900$

Also, $P = Rs.6750$

$\therefore \quad r = \frac{900 \times 100}{6750 \times 4} = \frac{10}{3}$

$r = 3\frac{1}{3}\%\ p.a$

i. $e.$., which is the required rate.

Example 6. ***A sum of*** $Rs.\ 720$ ***is due after*** $4\ \ years$ ***at*** $5\%\ \ p.\ \ a.$ ***simple interest. Find its present worth*** **?**

Solution : Let 'P' be the required present worth.

Now, according to given question

$A = Rs.720$

$t = 4\ years$

$r = 5\%\ p.a.$

As $\Rightarrow$ $A = P + S.I.$

i. $e.$ $A = P + \frac{P \times r \times t}{100}$

$= P\left(1 + \frac{rt}{100}\right)$

$720 = P\left(1 + \frac{5 \times 4}{100}\right)$

$720 = P\left(1 + \frac{20}{100}\right)$

$\Rightarrow$

$\Rightarrow$

$\Rightarrow \quad 720 = P\,(\frac{120}{100})$

$P = \frac{720 \times 100}{120} = Rs.\,600$

$\Rightarrow$

Hence, the present worth of the given sum is $Rs.\,600$.

6.4 COMPOUND INTEREST

In this case, the borrower and the lender agree to fix up a certain interval of time (that may be a year or a half-year or a quarter of year or a month etc.), so that the amount (i. $e.$ $Principal + Interest$) at the end of each interval becomes the new principal for the next interval, thus increasing the principal base, at the end of each specified period of time, on which subsequent interest is calculated. By repeating this process till the last period of specified time, if we calculate the total interest over all the intervals, the interest thus calculated is called the compound interest.

i. $e.$ $\quad C.I. = Amount - Principal$

i. $e.$ the difference between the final amount and the original principal gives required compound interest for the given time.

Note :

Interest need not to be compounded annually only, but it may be compounded semi-annually,
i. $e.$ two times a year or quarterly i. $e.$ four times a year, and so on. This fixed interval of time at the end of which the interest is calculated and added to the principal to form a new principal for the next interval is called the conversion period.

The formula for calculating the compound amount and compound interest is given below:

$$A = P\,(1 + \frac{r}{100})^{t}$$

Where $P =$ Principal

$r =$ Rate of Interest per payment period

$t =$ Number of payment periods or total number of conversion periods

$A =$ Total amount accumulated after t payment periods.

$\therefore$ $Compond\ Interest = Amount - Principal$

i. $e.$ $C.I. = A - P$

i. $e.$ $C.I. = P\left[\left(1 + \frac{r}{100}\right)^t - 1\right]$

Note :

If r_1, r_2, r_3 are the rates of interest (per payment period) for first n_1 conversion periods, second n_2 conversion periods, third n_3 conversion periods then

$$A = P\left(1 + \frac{r_1}{100}\right)^{n_1}\left(1 + \frac{r_2}{100}\right)^{n_2}\left(1 + \frac{r_3}{100}\right)^{n_3}$$

DOUBLING PERIOD

The time period in which an amount becomes double of itself at a certain given rate of interest can be calculated by putting $A = 2P$ and then finding t.

This doubling period can also be calculated by using the following formula:

(a) Rule of 72 :

$$Doubling\ Period = \frac{72}{Rate\ of\ interest}$$

(b) Rule of 69:

$$Doubling\ Period = 0.35 + \frac{69}{Rate\ of\ interest}$$

Example 7. ***Find the compound interest on*** $Rs.\ 500$ ***for*** $3\ years$ ***at*** $5\%\ p.\ a.$ ***compound annually.***

Solution : Here, $P = Rs.\ 500,\ t = 3\ years,\ r = 5\%\ p.\ a.$

$\therefore \quad A = P\left(1 + \frac{r}{100}\right)^t$

$$= 500\left(1 + \frac{5}{100}\right)^3$$

$$= 500(1.05)^3 = Rs. 578.8125$$

Hence, compound interest

$$C.I. = A - P$$

$$= 578.8125 - 500$$

$$= Rs. 78.81$$

Example 8. ***Mr. Ramesh deposits*** $Rs.\ 10,\ 000$ ***today at*** $6\%\ p.\ a.$ ***rate of interest. In many years will this amount become double of itself ? Work out this problem by using the rule of*** 72 ***and rule of*** 69**.**

Solution : According to rule of 72

$$Doubling\ period = \frac{72}{Rate\ of\ interest}$$

$$= \frac{72}{6} = 12\ years$$

and, According to rule of 69

$$Doubling\ period = 0.35 + \frac{69}{Rate\ of\ interest}$$

$$= 0.35 + \frac{69}{6}$$

$$= 0.35 + 11.50$$

$$= 11.85 \approx 12\ years.$$

Example 9. ***Find the compound amount of*** $Rs. 5000$ ***lent for*** $3\ years$ ***at*** $16\%\ p.a.$ ***converted.***

(a) Annually ***(b) Semi-Annually*** ***(c) Quarterly***

Solution : Here, $P = Rs. 5000$

$$t = 3\ Years$$

$$= 6\ half - years$$

$= 12\ quarters\ r$

$= 8\%\ p.\ a.$

$= 4\%\ half\ yearly$

$= 2\%\ quarterly$

Now,

(a) Annually

$$A = P\left(1 + \frac{r}{100}\right)^t$$

$$= 5000\left(1 + \frac{8}{100}\right)^3$$

$$= 5000\,(1.08)^3$$

$$= Rs.\,6298.56$$

(b) Semi-Annually

$$A = P\left(1 + \frac{r}{100}\right)^t$$

$$= 5000\left(1 + \frac{4}{100}\right)^6$$

$$= 5000\,(1.04)^6$$

$$= Rs.\,6326.59$$

(c) Quarterly

$$A = P\left(1 + \frac{r}{100}\right)^t$$

$$= 5000\left(1 + \frac{2}{100}\right)^{12}$$

$$= 5000\,(1.02)^{12}$$

$$= Rs.\,6341.20$$

Example 10. ***When a child was born, a sum of*** $Rs.\ 5000$ ***was deposited in his bank account. The bank pays*** $12\%\ p.\ a.$ ***interest compounded half-yearly. What amount would be received by the child on his*** 5^{th} ***birthday*** **?**

Solution : Here, the amount received on child's fifth birthday will be the future value of $Rs. 5000$ after 5 $years$

Now, $P = 5000$

$r = 12\% \ p.a.$

$= 6\% \ half \ yearly$

$t = 5 \ years$

$= 10 \ half \ years$

$\therefore \quad A = P\left(1 + \frac{r}{100}\right)^t$

$= 5000\left(1 + \frac{6}{100}\right)^{10}$

$= 5000(1.06)^{10}$

$= Rs. 8954$

Example 11. ***Find the compound interest on*** $Rs. \ 7000$ ***for*** 3 $years$***, if the rate of interest for first two years is*** $6\% \ p.a.$ ***and*** $9\% \ p.a.$ ***for the third year half yearly.***

Solution : Here, $P = Rs. 7000$

r_1 = rate of interest for first two years $= 6\% \ p.a.$

r_2 = rate of interest for the third year $= 9\% \ p.a. = \frac{9}{2}\% \ half \ yearly$

$t = t_1 + t_2$

$= 2 \ years + 1 \ year$

$= 2 \ years + 2 \ half \ years$

Now,

$$Compound\ Amount = Principal + Compound\ Interest$$

$\Rightarrow$ $A = P + C.I.$

$C.I. = A - P$

$$= P\left(1 + \frac{r_1}{100}\right)^{t_1}\left(1 + \frac{r_2}{100}\right)^{t_2} - P$$

$$= 7000\left[\left(1 + \frac{6}{100}\right)^2\left(1 + \frac{9}{200}\right)^2 - 1\right]$$

$\Rightarrow$

$$= 7000\,[(1.06)^2(1.045)^2 - 1]$$

$$7000\,[(1.1236)(1.092025) - 1]$$

$$7000\,[0.2269929]$$

$$Rs.\,1588.99$$

$$Rs.\,1589$$

=

=

=

≈

Example 12. ***In how many years will a certain sum double itself at*** $10\%\ p.\ a.$ ***when the interest is compounded half yearly*****.**

Solution : Here, if 'P' is the principal amount, then

$$A = 2P$$

Also, $r = 10\%\ p.a. = 5\%\ half\ yearly$

$t = ?$, where $t =$ no. of half years

Now, as

$$A = P\left(1 + \frac{r}{100}\right)^t$$

$\therefore$ $2P = P\left(1 + \frac{5}{100}\right)^t$

$$2 = \left(\frac{105}{100}\right)^t$$

$$2 = \left(\frac{21}{20}\right)^t$$

$\Rightarrow$

$\Rightarrow$

Taking log on both sides, we get

$$\log 2 = t(\log 21 - \log 20)$$

$\Rightarrow$ $0.3010 = t(1.3222 - 1.3010)$

$$t(0.0212) = 0.3010$$

$$t = \frac{3010}{212} = 14.2\ half\ years$$

$\Rightarrow$

$\Rightarrow$

$\therefore$ Required time $= \frac{14.2}{2} = 7.1\ years$

Example 13. ***At what rate of compound interest will a certain sum double itself in $6\ years$?***

Solution : Here, if 'P' is the principal amount, then

$$Compound\ Amount, A = 2P$$

Also, $t = 6\ years$

$r = ?$

$\therefore$ $A = P\left(1 + \frac{r}{100}\right)^t$

$$2P = P\left(1 + \frac{r}{100}\right)^6$$

$$\frac{2P}{P} = \left(1 + \frac{r}{100}\right)^6$$

$$2 = \left(1 + \frac{r}{100}\right)^6$$

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$

Taking log on both sides, we get

$\log 2 = 6 \log(1 + \frac{r}{100})$

$\Rightarrow \quad \frac{0.3010}{6} = \log(1 + \frac{r}{100})$

$0.0501 = \log(1 + \frac{r}{100})$

$\text{Antilog}(0.0501) = (1 + \frac{r}{100})$

$1 + \frac{r}{100} = \text{Antilog}(0.0501)$

$1 + \frac{r}{100} = 1.122$

$\frac{r}{100} = 1.122 - 1 = 0.122$

$r = 0.122 \times 100 = 12.2$

$\Rightarrow$

$\Rightarrow$

i. e.

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$

$\therefore$ Required rate of interest $= 12.2\%\ p.\ a.$

6.5 TEST YOUR UNDERSTANDING (A)

1. Find the simple interest on a sum of $Rs.\ 600$ for 4 years at $5\%\ p.\ a.$

2. At what rate percent a sum of $Rs.\ 5000$ amounts to $Rs.\ 10{,}000$ in 5 years.

3. How long will it take for a certain sum of $Rs.\ 2000$ to accumulate to $Rs.\ 2500$ at $5\%\ p.\ a.$

4. A certain sum amounts to $Rs.\ 2784$ in 2 years and $Rs.\ 3360$ in 5 years at simple interest. Find the sum and the rate of simple interest.

5. Calculate the compound value of $Rs.\ 12{,}000$ at the end of 3 years at $10\%\ p.\ a.$ rate of interest. Also find the compound interest.

6. Find the amount of $Rs.\ 2800$ lent for the $1\frac{1}{2}$ years at $10\%\ p.\ a.$, interest being payable compounded half yearly.

7. Ram offers his friend to pay $Rs.\ 12,000$ after 15 years in exchange of $Rs.\ 1000$ today. What interest rate is implicit in the offer ?

8. Divide $Rs. 24,000$ between Ram and Shyam so that the amount Ram receives in 4 years at $12.5\%\ p.\ a.$ rate of interest is equal to the amount received by Shyam in 3 years at $10\%\ p.\ a.$ rate of interest.

9. In how many years will a certain sum double itself at $10\%\ p.\ a.$, interest being compounded quarterly.

10. The difference between compound interest and simple interest on a certain sum of money at $5\%\ p.a.$ for 2 years is $Rs. 2.30$. Find the sum ?

11. An investment company pays $12\%\ p.\ a.$ rate of interest by compounding the sum quarterly. If $Rs. 5000$ is deposited initially, how much shall it grow to in 5 years ?

12. What rate of interest per annum doubles an investment in 7 years, if the interest is being compounded annually ?

Answers

1. $Rs. 120$

2. 20%

3. 5 years

4. $Rs. 2400$, 8%

5. $Rs. 15,972$, $Rs. 3972$

6. $Rs. 3241.35$

7. 18%

8. $Rs. 10,892$, $Rs. 13,108$

9. 7.03 years

10. $Rs. 1000$

11. $Rs. 9030$

12. $10.41\%\ p.a.$

6.6 NOMINAL AND EFFECTIVE RATE OF INTEREST

From the above discussion and examples it is clear that in case of compound interest if the interest is being compounded more than once a year, the amount grows faster. It is so because the actual rate of interest realized, called *effective rate*. In case of multi-period compounding is more than the apparent annual rate of interest called *nominal rate*.

For example, the future value of $Rs.\ 100$ at the end of one year, at the rate of interest $10\%\ p.\ a.$ will become $Rs.\ 110$. But if the interest is calculated on half yearly basis, then At the end of first six months, we will have $Rs. 105$.

i. $e.,\ Rs.\ 100 + 5\%$ of $Rs.\ 100 = Rs.\ 105$ And so,
at the end of next six months, it will be

$$Rs. 105 + 5\% \text{ of } Rs. 105 = Rs. 110.25$$

Hence, the total interest realized in case half yearly compounding is $Rs.\ 10.25$ or we can say that effective rate of interest is 10.25% while the nominal rate is 10%.

6.6.1 NOMINAL RATE OF INTEREST : In case of compound interest, the apparent annual rate of interest is called the *nominal rate of interest*.

6.6.2 EFFECTIVE RATE OF INTEREST : In case of compound interest, if the interest is being calculated more than once a year (i. $e.,$ a case of multi-period compounding) the interest actually earned or realized is called the *effective rate of interest*.

Relationship between Nominal and Effective Rate of Interest

Let, $E =$ Effective rate of interest $r =$ Nominal rate of interest

$=$ Frequency of compounding per year

Then,

$$E = \left(1 + \frac{r}{100m}\right)^m - 1$$

Example 14. ***Find the effective rate of interest equivalent to nominal rate of*** 6% ***compounded monthly.***

Solution : $r = 6\%\ p.a.$

$m = 12$ Now,

as we have

$$E = \left(1 + \frac{r}{100m}\right)^m - 1$$

∴ Effective rate of interest,

$$E = \left(1 + \frac{6}{1200}\right)^{12} - 1$$

$$= (1.0075)^{12} - 1$$

$$= 1.0930 - 1$$

$$= 0.0930$$

$$= 9.3\%$$

Example 15. ***Which is better option for an investor, compounding*** $9.\ 1\%$ ***semi-annually or*** 9% ***monthly*** **?**

Solution : Let us find the effective rate of interest when compounding is done at 9.1% semi-annually.

∴ For the first investment,

$r = 9.1\%\ p.a.$

$m = 2$

∴ $E = \left(1 + \frac{9.1}{200}\right)^2 - 1$

$$= (1 + 0.0455)^2 - 1$$

$$= 0.09307025 = 9.30\%$$

i. $e.,$ Effective rate of interest in case of first investment $= 9.30\%$

Now, to find the effective rate of interest when compounding is done at 9% monthly, we have

$r = 9\% \; p.a.$

$m = 12$

$\Rightarrow \quad E = \left(1 + \frac{9}{1200}\right)^{12} - 1$

$= (1 + 0.0075)^{12} - 1$

$x = (1.0075)^{12}$ ...(i)

Let

Taking log on both sides, we get

$$\log x = 12 \log(1.0075)$$
$$= 12(0.0034)$$

$\therefore \quad \log x = 0.0408$

$\Rightarrow \quad x = \text{Antilog}\ (0.0408)$

$\Rightarrow \quad x = 1.098$

$\therefore$ (i) becomes

$$E = 1.098 - 1 = 0.098 = 9.8\%$$

i. $e.$, Effective rate of interest in case of second investment = 9.8%

$\therefore$ Second investment option is better

6.7 CONTINUOUS COMPOUNDING OF INTEREST

We know that in case of compound interest, the compound amount 'A' for the principal 'P' after 't' year at rate of r % $p.a.$,

$$\text{Amount}, A = P\left(1 + \frac{r}{100}\right)^{t}$$

But, if the compounding is done 'm' times in a year, then we will have

$$A = P\left(1 + \frac{r}{m \times 100}\right)^{t \times m}$$

Where, $\quad m =$

When the compounding is done continuously i. $e.$, m becomes very large($m \to \infty$), we have

$$A = \lim_{m \to \infty} P\left(1 + \frac{r}{m \times 100}\right)^{t \times m}$$

$$= P\left[\lim_{m \to \infty}\left(1 + \frac{r}{100m}\right)^{\frac{100m}{r}}\right]^{\frac{rt}{100}}$$

$$= Pe^{rt/100}$$

As $\left(\lim_{x \to \infty}(1 + x)^{1/x} = e\right)$

Where e is the numerical constant and $e = 2.7183$

Thus, the compound amount 'A' accumulated after 't' years, if the interest is compounded continuously at the rate 'r', is

$$A = Pe^{rt/100}$$

Example 16. ***Which investment option is better*** 16% ***compounded quarterly or*** $12.\ 2\%$ ***compounded continuously*** **?**

Solution : For first investment Let

$P = Rs.\,100$

$r = 16\%\ p.a. = \frac{16}{4}\%$ quarterly

$= 4\%$ quarterly

$t = 1$ year $= 4$ quarters

$\therefore\ A = 100\left(1 + \frac{4}{100}\right)^4 = 100(1.04)^4$

$$= 100(1.169858)$$

$$= Rs. 116.98$$

⇒ Effective rate of interest

$$= (116.98 - 100)$$

$= 16.98\%$ For

second investment

Let $P = Rs. 100$

$$r = 12.2\% \ p.a.$$

$$t = 1 \ year$$

As interest is being compounded continuously

∴ $A = Pe^{rt/100}$

⇒ $A = 100 \times e^{12.2/100} = 100e^{0.122}$

Taking log on both sides, we get

$$\log A = \log 100 + 0.122 \log e$$

$$= 2 + (0.122) \log(2.7183) \qquad (\because e = 2.7183)$$

$$= 2 + (0.122)(0.4343)$$

$$= 2 + 0.053$$

$$= 2.053$$

⇒ $A = \text{Antilog}(2.053) = 113.0$

⇒ $A = Rs. 113.0$

Thus, Effective rate of interest

$$= (113.0 - 100)$$
$$= 13\%$$

∴ First investment is better

Note. We can also say that if the nominal rate r is compounded continuously, then the formula of effective rate E, becomes

In case of continuous compounding

$$E = (e^{r/100} - 1)$$

or $E = 100(e^{r/100} - 1)\%$

Example 17. ***Find the effective rate equivalent to nominal rate*** 6% ***compounded***

(a) Monthly ***(b) Continuously***

Solution : (a) We have

$$r = 6\%$$

$$m = 12$$

∴ Effective rate, $E = \left(1 + \frac{r}{1200}\right)^{12} - 1$

$$= (1.005)^{12} - 1$$

$$= 1.0616 - 1$$

$$= 0.0616$$

$$= 6.16\%$$

(b) We have

$$r = 6\%$$

∴ $E = e^{r/100} - 1$

$$= e^{6/100} - 1$$

$$= e^{0.06} - 1 \quad \dots(i)$$

Take $x = e^{0.06}$

∴ $\log x = 0.06 \log e = 0.06 \times \log(2.7183)$

⇒ $\log x = 0.06 \times 0.4343$

$$= 0.026058$$

⇒ $x = \text{Antilog}\ (0.0260)$

∴ $x = 1.062$

⇒ (i) becomes

$$E = 1.062 - 1 = 0.062 = 6.2\%$$

⇒ $E = 6.2\%$

6.8 TEST YOUR UNDERSTANDING (B)

1. Find the effective rate equivalent to nominal rate 12% compounded quarterly.

2. Which is better investment either 5.7% compounded half yearly or 5% compounded monthly ?

3. Find the effective rate of interest corresponding to the nominal rate of interest 8% $p.\ a.$, if it is converted to

   (i) Semi-annually (ii) Quarterly

4. If you invest $s.$ 4,50,000 at an annual interest rate of 7% compounded continuously, then find the final amount you will have in the account after 5 years.

Answers

1. 12.55%
2. First
3. (i) 8.16% half yearly (ii) 8.24% quarterly
4. $Rs.$ 6,38,550

6.9 COMPOUNDING & DISCOUNTING OF A SUM USING DIFFERENT TYPE OF RATES

6.9.1 COMPOUNDING (OR GROWTH)

If 'r' is the rate of increase per 100 per year, then the value after 'n' years (P_n) is given by

$$P_n = P_o \left(1 + \frac{r}{100}\right)^n$$

Where P_o = Present Value

If the rate of increases is different for different years, i. $e.$ $r_1, r_2, \ldots, r_n$ then (P_n) is given by

$$P_n = P_o \left(1 + \frac{r_1}{100}\right)\left(1 + \frac{r_2}{100}\right)\ldots\left(1 + \frac{r_n}{100}\right)$$

Example 18. ***The present population of a town is*** 2, 80, 000***. If it increase at the rate of*** 6% $p.\ a.$ ***What will be its population after*** 2 ***years*** **? Solution :** Population after 'n' years, i. $e.$

$$P_n = P_o\left(1+\frac{r}{100}\right)^n$$

Where P_o = Present Population

= 2,80,000

r = Rate of Growth

= 6% $p.\ a.$

and n = Time = 2 years

⇒ Population after 2 years

$$= 2{,}80{,}000\left(1+\frac{6}{100}\right)^2$$

$$= 2{,}80{,}000 \times \frac{106}{100} \times \frac{106}{100}$$

$$= 3{,}14{,}608$$

Example 19. ***The present population of a tour is*** 50, 000***. It grows at the rate of*** 4%***,*** 5% ***and*** 8% ***during first year, second year and third year respectively. Find its population after*** 3 ***years*** **?**

Solution : If 'P' denotes the population of the tour after 3 years, then

$$P = P_o\left(1+\frac{r_1}{100}\right)\left(1+\frac{r_2}{100}\right)\left(1+\frac{r_3}{100}\right)$$

Where $P_o = 50{,}000$,

$r_1 = 4\%$,

$r_2 = 5\%$ and

$r_3 = 8\%$

$$\therefore \quad = 50{,}000\left(1+\frac{4}{100}\right)\left(1+\frac{5}{100}\right)\left(1+\frac{8}{100}\right)$$

$$= 50{,}000 \times \frac{104}{100} \times \frac{105}{100} \times \frac{108}{100}$$

$$= 50{,}000 \times \frac{26}{25} \times \frac{21}{20} \times \frac{27}{25}$$

$$= 61{,}740$$

6.9.2 DISCOUNTING (ON DEPRECIATION OR PRESENT VALUE)

The value of certain fixed assets like plant, machinery etc. decrease (due to wear and tear) with the passage of time. This relative decrease in the value of fixed assets over a certain period of time is called *depreciation*. This process is called *Discounting*.

If '$r\%$' is the rate of depreciation per annum, then the value of the fixed asset after 'n' years i. $e.$, V_n

$$V_n = V_o \left(1 - \frac{r}{100}\right)^n$$

Where, V_0 = The value of the fixed asset at present

If the rate of depreciation is different for different years, i. $e.$ $r_1, r_2, \ldots, r_n$ then (V_n) is given as

$$_n = V_o \left(1 - \frac{r1}{100}\right)\left(1 - \frac{r2}{100}\right)\ldots\left(1 - \frac{rn}{100}\right)$$

Example 20. ***The value of a machine depreciates at*** 12% ***annually. If its present value is*** $Rs.\ 38, 720$***. What will its value after*** 6 ***years*** **?**

Solution : The value of the machine after 6 years V_6 is given by

$$V_6 = V_o \left(1 - \frac{r}{100}\right)^n$$

Where V_o = Present value of the machine

$= Rs.\ 38{,}720$

r = Annual rate of depreciation

$= 12\%$

n = Time period

$= 6$ years

$$V_6 = 38{,}720 \left(1 - \frac{12}{100}\right)^6$$

$$= 38{,}720 \left(\frac{88}{100}\right)^6$$

∴

Taking log on both sides, we get

$$\log V_6 = \log 38{,}720 + 6(\log 88 - \log 100)$$

$$= 4.5879 + 6(1.9445 - 2.0000)$$

$$= 4.5879 + 6(-0.0555)$$

$= 4.5879 - 0.3330$
$= 4.2549$

$\Rightarrow \quad V_6 = \text{Antilog}(4.2549) = 17{,}990.0$

$\therefore$ Depreciated value of the machine $= Rs.\ 17{,}990$

Example 21. ***The initial cost of a machine is*** $Rs.\ 64,\ 000$***. For the first*** 2 ***years, the rate of depreciation is*** 5%***, then for the next*** 2 ***years it becomes*** 8% ***and for the fifth year it depreciates at*** 10%***. Find the depreciated value of the machine after*** 5 ***years*** **?**

Solution : The depreciated value of the machine after 5 years, i.e. V_5 is given by

$$V_5 = V_o\left(1 - \frac{r_1}{100}\right)^{n_1}\left(1 - \frac{r_2}{100}\right)^{n_2}\left(1 - \frac{r_3}{100}\right)^{n_3}$$

Where $\quad V_o =$ Initial cost of the machine

$r_1 = 5\%\ p.a.$ and $n_1 = 2$

$r_2 = 8\%\ p.a.$ and $n_2 = 2$

$r_3 = 10\%\ p.a.$ and $n_3 = 1$

$$V_5 = 64{,}000\left(1 - \frac{5}{100}\right)^2\left(1 - \frac{8}{100}\right)^2\left(1 - \frac{10}{100}\right)^1$$

$$= 64{,}000\left(\frac{95}{100}\right)^2\left(\frac{92}{100}\right)^2\left(\frac{90}{100}\right)^1$$

$$= 64{,}000 \times \frac{19}{20} \times \frac{19}{20} \times \frac{23}{25} \times \frac{23}{25} \times \frac{9}{10}$$

$$= 43{,}999.26$$

$\therefore$

$\therefore$ Depreciated cost of the machine $= Rs.\ 43{,}999.26$

Example 22. ***The value of a machine originally costing*** $Rs.\ 48,\ 000$ ***depreciates at*** $10\%\ p.\ a.$ ***and eventually its scrap value becomes*** $Rs.\ 18,\ 000$***. Estimate its effective life during which it remained in use.***

Solution : Let 'n' years be the effective life of the machine. Then, we have

$V_o =$ Original cost of machine $= Rs.\ 48{,}000$

$r =$ Rate of depreciation $= 10\%\ p.a.$

$V_n =$ Scrap value of the machine after 'n' years $= Rs.\ 18{,}000$

$$V_n = V_o \left(1 - \frac{r}{100}\right)^n$$

$$18{,}000 = 48{,}000 \left(1 - \frac{10}{100}\right)^n$$

So,

⇒

∴ $\frac{18}{48} = \left(\frac{90}{100}\right)^n$

$\frac{3}{8} = \left(\frac{9}{10}\right)^n$

⇒

Taking log on both sides, we get

⇒ $\log\left(\frac{3}{8}\right) = \log\left(\frac{9}{10}\right)^n$

$\log 3 - \log 8 = n(\log 9 - \log 10)$

$- 0.9031 = n(0.9542 - 1)$

$-0.4260 = -0.0458 \times n$

$= \frac{0.4260}{0.0458} = \frac{4260}{458} = 9.3$

⇒

⇒ 0.4771

⇒ ⇒

∴ Machine remained in use for 9.3 years.

6.10 TEST YOUR UNDERSTANDING (C)

1. The present population of a town is 40,960. It increases at the rate of 6.25% $p.\ a.$ What will be its population after 2 years ?

2. The population of a town is $2,50,000$ at present. It increases at 4%, 5% and 8% during first year, second year and third year respectively. Find its population after 3 years ?

3. A population grows at the rate of 2.3% per year. How long does it take for the population to double ?

4. The value of a car depreciates at 9% annually. If its present value is $Rs.\ 3,75,000$. What will be its value after 6 years ?

5. The value of a machinery plant depreciates by 12% annually. It its present value is $Rs.$ $38,720$. Find its value 2 years ago.

6. An electric typewriter worth $Rs.\ 12,500$ depreciates at the rate of $12\%\ p.\ a.$ Ultimately it was sold for $Rs.\ 9680$. Estimate the effective life during which the electronic typewriter remained in use.

7. A certain machinery depreciates at the rate of $8\%\ p.\ a.$ How long does it take for the value of the machinery to reduce to $\frac{1}{4}$ of its original value ?

Answers

1. $49,130$

2. $2,94,840$

3. 31 Years (approx.)

4. $Rs.$ $2,12,950$

5. $Rs.$ $49,990$

6. 2 Years

7. 16.63 Years (approx.)

6.11 LET US SUM UP

- Interest is the extra amount paid by borrower to the lender on the funds provided by him.

- Interest can be simple interest or compound interest. • Simple interest is charged on Principal amount only.
- Compound interest is charged on Principal amount plus accumulated Interest.
- Nominal interest is the interest apparent on the amount.
- Real Interest is the interest actually earned by the lender.
- Compounding can be done on periodic basis or on continuous basis.
- From present value finding future value is called compounding and from future value finding present value is called discounting.

6.12 KEY TERMS

- **INTEREST:**Interest is the additional amount paid by a borrower for the use of a lender's money.
- **SIMPLE INTEREST:** Simple interest is the interest computed only on the original amount borrowed (or lent). It is the extra money paid (or taken) on that principal for one time period.
- **COMPOUND INTEREST**: Compound interest is charged on Principal amount plus accumulated Interest.
- **NOMINAL RATE OF INTEREST :** In case of compound interest, the apparent annual rate of interest is called the *nominal rate of interest*.
- **EFFECTIVE RATE OF INTEREST :** In case of compound interest, if the interest is being calculated more than once a year (i. *e.,* a case of multi-period compounding) the interest actually earned or realized is called the *effective rate of interest*.

6.13 FURTHER READINGS

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B. COM (DIGITAL)

SEMESTER II

COURSE: BUSINESS MATHEMATICS AND STATISTICS

UNIT 7 – MEASURES OF CENTRAL TENDENCY

STRUCTURE

7.0 OBJECTIVES

After studying the Unit, students will be able to

- Define what Average.
- Know why average is calculated.
- Understand features of good measure of Average.
- Find different types of Averages for various types of data.
- Understand the relation that exists between different types of Averages. • Know merits and limitations of each type of average.

7.1 INTRODUCTION

We can say that modern age is the age of Statistics. There is no field in the modern life in which statistics is not used. Whether it is Business, Economics, Education. Government Planning or any other field of our life, statistics is used everywhere. Business manager use statistics for business decision making, Economists use statistics for economic planning, Investors use statistics for future forecasting and so on. There are many techniques in statistics that helps us in all these purposes. Average or Central Tendency is one such technique that is widely used in statistics. This technique is used almost in every walk of the life.

7.2 MEANING OF AVERAGE OR CENTRAL TENDENCY

Average or Central tendency is the most used tool of statistics. This is the tool without which statistics is incomplete. In simple words we can say that Average is the single value which is capable of representing its series. It is the value around which other values in the series move. We can define Average as the single typical value of the series which represent whole data of the series. Following is the popular definition of average:

"An average is a single value within the range of data that is used to represent all values in the series. Since an average is somewhere within the range of the data, it is also called a measure of Central Value". ***: -Croxton and Cowden***

7.3 OBJECTIVES AND FUNCTIONS OF AVERAGE

1. **SINGLE VALUE REPRESENTING WHOLE DATA:** In statistics data can be shown with the help of tables and diagrams. But some time data is very larger and it is not easy to present in table or graph. So, we want to represent that data in summarised form. Average helps us to represent data in summarised form. For example, that data of national income of India is very large but when we calculate per capita income it gives us idea of the national income.
2. **TO HELP IN COMPARISON**: In case we want to compare two different series of data, it is very difficult to compare. There are many difficulties like number of items in the series may be different. In such case average helps us in making the comparison. For example, if we want to compare income of people living in different countries like India and Pakistan, we can do so by calculating per capita income which is a form of average.
3. **DRAW CONCLUSION ABOUT UNIVERSE FROM SAMPLE:** This is one of the important function of average. If we take the average of a sample, we can draw certain conclusion about the universe from such Average. For example mean of a sample is representative of its universe.
4. **BASE OF OTHER STATISTICAL METHODS:** There are many Statistical

Techniques that are based on average. If we don't have an idea about the average, we cannot apply those techniques. For example Dispersion, Skewness, Index Number are based on average.

5. **BASE OF DECISION MAKING**: Whenever we have to make certain decision, average plays very crucial role in the decision making. From the average we could have idea about the data and on the basis of that information we can take decision. For example, a company can take decision regarding its sales on the basis of average yearly sales of past few years.
6. **PRECISE RELATIONSHIP**: Average helps us to find out if there is precise relation between two variables or two items. It also removes the biasness of the person making analysis. For example if you say that Rajesh is more intelligent than Rav,i it is only our personal observation and does not make any precise relation. If we compare the average marks of both the students we could have a precise relation.
7. **HELPFUL IN POLICY FORMULATION**: Average helps the government in formulation of the policy. Whenever government has to formulate economic policy they consider various averages like per capita income, average growth rate etc.

7.4 REQUISITE / FEATURES OF GOOD AVERAGE

1. **RIGIDLY DEFINED:** A good measure of average is one which is having a clear cut definition and there is no confusion in the mind of person who is calculating the average. In case person applies his discretion while calculating the average, we cannot say that average is a good measure. Good average must have fix algebric formula, so that whenever average of same data is calculated by two different persons, result are always same.
2. **EASY TO COMPUTE**: Good average is one which does not involve much calculation and are easy to compute. A good average is one which can be calculated even by a person having less knowledge of Statistics. If it is very difficult to calculate the average, we cannot regard it as a good measure.
3. **BASED ON ALL OBSERVATIONS:** Good average must consider all the values or data that is available in the series. If average is based on only few observations of the series, we cannot say that it is a good measure of average.
4. **NOT AFFECTED BY EXTREME VALUES**: A good measure of average is one which is not affected by the extreme values present in the Data. Sometime data contains values which are not within normal limits, these values are called extreme values. If average is affected by these extreme values, we cannot claim that average is a good measure.
5. **REPRESENTATIVE OF WHOLE SERIES:** A good measure of average is one which represent characteristics of whole series of the data.
6. **EASY TO UNDERSTAND:** A good measure of average is one that is not only easy to understand but also easy to interpret.
7. **NOT AFFECTED BY FLUCTUATIONS IN THE SAMPLING:** If we take one sample from the universe and calculate average, then we draw another sample from the same universe and calculate the average again, there must not be much difference

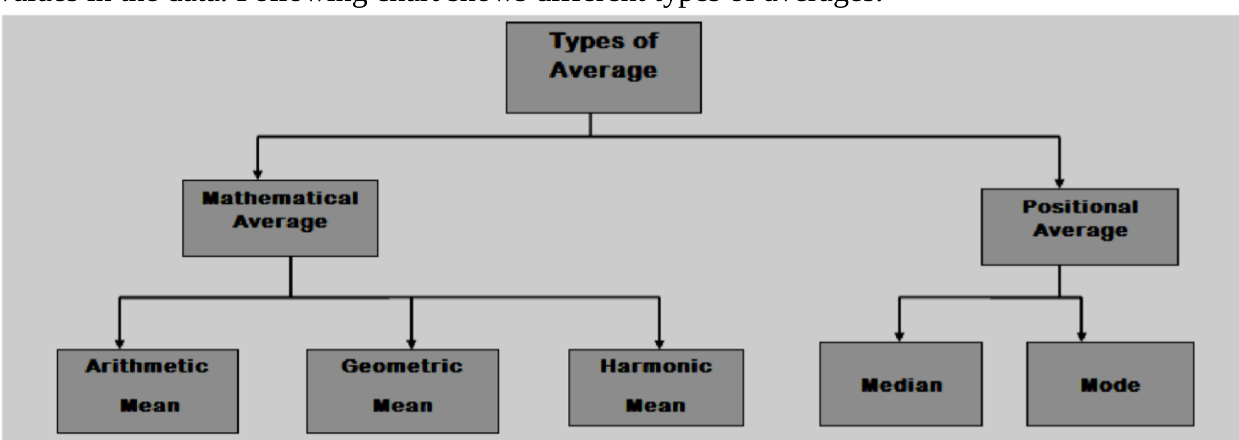
between these two averages. If average significantly change with the change in sample, we cannot treat it as a good measure of average.

8. **CAPABLE OF FURTHER ALGEBRAIC TREATMENT:** a good average is one on which we can apply further algebraic treatment. In case further algebraic treatment is not possible, we cannot say that it is a good average. Sch further algebraic treatment may be anything like calculating combined average when average of two different series is available.
9. **LOCATED GRAPHICALLY:** It will be better if we can locate average graphically also. Graphs are easy to understand and interpret, so the average that can be located graphically is a good average.

As no single average has all these features, we cannot say which measure of average is best. Each measure has its own merits and limitation. Moreover, each measure is suitable for particular situation.

7.5 MEASURES OF CENTRAL TENDENCY

There are many methods through which we can calculate average or central tendency. We can divide these methods into two categories that are Algebraic Method and Positional Average. Algebraic methods are those in which the value of average depends upon the mathematical formula used in the average. The mathematical average can further be divided into three categories that are Arithmetic Mean, Geometric Mean and Harmonic Mean. On the other hand positional average are those average which are not based on the mathematical formula used in calculation of average rather these depends upon the position of the variable in the series. As these depends upon the position of the variable, these averages are not affected by the extreme values in the data. Following chart shows different types of averages.



7.6 ARITHMETIC MEAN

It is the most popular and most common measure of average. It is so popular that for a common man the two terms Arithmetic Mean and Average are one and the same thing. However, in reality these two terms are not same and arithmetic mean is just one measure of the average. We can define the arithmetic mean as:

" The value obtained by dividing sum of observations with the number of observations".

So arithmetic mean is very easy to calculate, what we have to do is just add up the value of all the items given in the data and then we have to divide that total with the number of items in the data. Arithmetic mean is represented by symbol $A.M.\ or\ \overline{X}$

7.6.1 ARITHMETIC MEAN IN CASE OF INDIVIDUAL SERIES

Individual series are those series in which all the items of the data are listed individually. There are two methods of finding arithmetic mean in the individual series. These two methods are Direct method and Shortcut Method.

1. **DIRECT METHOD:** According to this method calculation of mean is very simple and as discussed above, we have to just add the items and then divide it by number of items.

Following are the steps in calculation of mean by direct method:

1. Suppose our various items of the data are X_1, X_2, X_3 .... X_n 2. Add all the values of the series and find $\sum X$.
3. Find out the number of items in the series denoted by n.
4. Calculate arithmetic mean dividing sum value of observation with the number of observations using following formula:

$$\overline{X} = \frac{X1 + X2 + X3 + --------Xn}{} = \frac{\sum X}{N}$$

Where $\overline{X} = Mean$

N = Number of items

$\sum X$ = Sum of observation

Example 1. ***The daily income of*** 10 ***families is a as given below (in rupees) :***

130, 141, 147, 154, 123, 134, 137, 151, 153, 147 ***Find the arithmetic mean by direct method*.**

Solution : Computation of Arithmetic Mean

Serial No.	Daily Income (in *Rs.*) X
1	130
2	141
3	147
4	154
5	123

6	134
7	137
8	151
9	153
10	147
$N = 10$	$\sum X = 1417$

$$A.M., \quad \overline{X} = \frac{X1 + X2 + X3 + -------XN}{N} = \frac{\sum X}{N} = \frac{1417}{10} = Rs. 141.7$$

2. **SHORT CUT METHOD:**Normally this method is used when the value of items is very large and it is difficult to make calculations. Under this method we take one value as mean which is known as assumed mean and deviations are calculated from this as you mean. This method is also known as is assumed mean method. Following are the steps of this method:
   1. Suppose our various items of the data are X_1, X_2, X_3 .... X_n
   2. Take any value as assumed mean represented by 'A'. This value may be any value among data or any other value even if that is not presented in data.
   3. Find out deviations of items from assumed mean. For that deduct Assumed value from each value of the data. These deviations are represents as 'dx'
   4. Find sum of the deviations represented by ∑dx.
   5. Find out the number of items in the series denoted by n.
   6. Calculate arithmetic mean dividing sum deviations of the observation with the number of observations using following formula:

$$\overline{X} = A + \frac{\sum dx}{N}$$

Where $\overline{X} = Mean$

A = Assumed Mean

N = Number of items

∑dx = Sum of deviations

Example 2. *Calculate* $A . M.$ ***by short - cut method for the following data***

R. No.	1	2	3	4	5	6	7	8	9	10
Marks	50	60	65	88	68	70	83	45	53	58

Solution : Let assumed Mean (A) be 60

R. No.	Marks (X)	$dx = X - A$
1	50	-10
2	60	0
3	65	5
4	88	28
5	68	8
6	70	10
7	83	23
8	45	-15
9	53	-7
10	58	-2
$N = 10$		$\sum dx = 40$

As $\overline{X} = A + \frac{\sum dx}{N}$

$\overline{X} = 60 + \frac{40}{10}$

$= 60 + 4$

⇒

⇒ $\overline{X} = 64$ Marks

7.6.2 ARITHMETIC MEAN IN CASE OF DISCRETE SERIES

In individual series if any value is repeated that is shown repeatedly in the series. It makes series lengthy and make calculation difficult. In case of discrete series, instead of repeatedly showing the items we just group those items and the number of time that item is repeated is shown as frequency. In case of discrete series we can calculate Arithmetic mean. By using Direct Method and Shortcut Method.

1. **DIRECT METHOD:** In indirect method we multiply the value of items (X) with their respective frequency (f) to find out the the product item (fX). Then we take up sun of the product and divide it with the number of items. Following are the steps
   1. Multiply the value of items (X) with their respective frequency (f) to find out the the product item (fX)
   2. Add up the product so calculated to find ∑fX.
   3. Find out the number of items in the series denoted by n.
   4. Calculate arithmetic mean dividing sum of the product with the number of observations using following formula:

$$\bar{X} = = \frac{\sum fX}{N}$$

Where $\bar{X} = Mean$

N = Number of items ∑fX = Sum of product of observations.

Example 3. ***Find the average income***

Daily Income (in rupees)	200	500	600	750	800
No. of Workers	2	1	4	2	1

Solution :

Daily Income ($Rs.$) X	No. of Workers Frequency (ƒ)	ƒ
200	2	400
500	1	500
600	4	2400
750	2	1500
800	1	800
	$\sum f = 10$	$\sum fX = 5600$

∴ Average Income $\bar{X} = \frac{\sum fK}{\sum f}$

$= \frac{5600}{10}$

$= Rs. 560$

2. **SHORT CUT METHOD:** Under this method we take one value as mean which is known as assumed mean and deviations are calculated from this as you mean. Then average is calculated using assumed mean. Following are the steps of this method:
   1. Suppose our items of the data are 'X' and its corresponding frequency is 'f'.
   2. Take any value as assumed mean represented by 'A'.
   3. Find out deviations of items from assumed mean. For that deduct Assumed value from each value of the data. These deviations are represents as 'dx'
   4. Multiply the values of dx with corresponding frequency to find out product denoted by fdx
   5. Find sum of the product so calculated represented by ∑fdx.
   6. Find out the number of items in the series denoted by n.
   7. Calculate arithmetic mean dividing sum deviations of the observation with the number of observations using following formula:

$$\overline{x} = A + \frac{\sum f dx}{N}$$

Where $\overline{x} = Mean$

A = Assumed Mean

N = Number of items $\sum$fdx = Sum of product of deviation with frequency.

Example 4. ***From the following data find out the mean height of the students*.**

Height (in cms.)	154	155	156	157	158	159	160	161	162	163
No. of Students	1	6	10	22	21	17	14	5	3	1

Solution : Let the Assumed Mean (A) be 150

Height in cms. X	No. of students f	$dX = (X - A)$ $= X - 150$	fdX
154	1	4	4
155	6	5	30
156	10	6	60
157	22	7	154
158	21	8	168
159	17	9	153
160	14	10	140
161	5	11	55
162	3	12	36
163	1	13	13
	$\sum f = 100$		$\sum fdX = 813$

Applying the formula

$$\overline{x} = A + \frac{\sum fdK}{\sum f}$$

We get

$$\overline{x} = 150 + \frac{813}{100}$$

$$= 150 + 8.13$$

$$= 158.13$$

$\therefore$ Mean Height = 158.13 cm

7.6.3 ARITHMETIC MEAN IN CASE OF CONTINUOUS SERIES

Continuous series is also known as Grouped Frequency Series. Under this series the values of the observation are grouped in various classes with some upper and lower limit. For example classes like 10-20, 20-30, 30-40 and so on. In the class 10-20 lower limit is 10 and upper limit is 20. So all the observations having values between 10 and 20 are put in this class interval. Similar procedure is adopted for all class intervals. The procedure of calculating Arithmetic Mean is continuous series is just like discrete series except that instead of taking values of observations we take mid value of the class interval. The mid value is represented by 'm' and is calculated using following formula:

$$m = \frac{Lower\ Limit + Upper\ Limit}{2}$$

1. **DIRECT METHOD:** In indirect method we multiply the mid values (m) with their respective frequency (f) to find out the the product item (fm). Then we take up sun of the product and divide it with the number of items. Following are the steps

   1. Multiply the mid values (m) with their respective frequency (f) to find out the the product item (fm)
   2. Add up the product so calculated to find ∑fm.
   3. Find out the number of items in the series denoted by n.
   4. Calculate arithmetic mean by dividing sum of the product with the number of observations using following formula:

$$\bar{x} = \ = \frac{\sum fm}{N}$$

Where $\bar{x} = Mean$

N = Number of items ∑fm = Sum of product of observations of mean and frequencies.

Example 5. ***Calculate the arithmetic mean of the following data*** **:**

Class Intervals $C.I.$	$100 - 200$	$200 - 300$	$300 - 400$	$400 - 500$	$500 - 600$	$600 - 70$
f	4	7	16	20	15	8

Solution :

Class Intervals $C.I.$	Mid Value m	Frequency f	fm
$100 - 200$	150	4	600
$200 - 300$	250	7	1750
$300 - 400$	350	16	5600
$400 - 500$	450	20	9000
$500 - 600$	550	15	8250

$600-700$	650	8	5200
		$\sum f = 70$	$\sum fm = 30{,}400$

As $\bar{x} = \frac{\sum fm}{\sum f}$

$\bar{x} = \frac{30{,}400}{70}$

$= 434.3$

∴

2. **SHORT CUT METHOD:** This method of mean is almost similar to calculation in the discrete series but here the assumed mean is selected and then the deviation are taken from mid value of the observations. Following are the steps of this method:
    1. Calculate the Mid Values of the series represented by 'm'.
    2. Take any value as assumed mean represented by 'A'.
    3. Find out deviations of items from assumed mean. For that deduct Assumed value from mid values of the data. These deviations are represents as 'dm'
    4. Multiply the values of dm with corresponding frequency to find out product denoted by fdm
    5. Find sum of the product so calculated represented by ∑fdm.
    6. Find out the number of items in the series denoted by n.
    7. Calculate arithmetic mean dividing sum deviations of the observation with the number of observations using following formula:

$$\bar{x} = A + \frac{\sum fdm}{N}$$

Where $\bar{x} = Mean$

A = Assumed Mean

N = Number of items ∑fdm = Sum of product of deviation from mid values with frequency.

Example 6. ***Calculate the mean from the following data***

Daily Wages (Rs.)	0 – 100	100 – 200	200 – 300	300 – 400	400 – 500	500 – 600	600 – 700	700 – 800	800 – 900
No. of Workers	1	4	10	22	30	35	10	7	1

Solution : Let the assumed mean, $A = 150$

Daily Wages $(Rs.)\ C.I.$	No. of Workers f	Mid Value m	$dm = m - A$ $(\quad - 150)$	fdm
$0 - 100$	1	50	-100	-100
$100 - 200$	4	150	0	0
$200 - 300$	10	250	100	1000
$300 - 400$	22	350	200	4400
$400 - 500$	30	450	300	9000
$500 - 600$	35	550	400	14,000
$600 - 700$	10	650	500	5000
$700 - 800$	7	750	600	4200
$800 - 900$	1	850	700	700
	$\sum f = 120$			$\sum fdm = 38{,}200$

As $\bar{x} = A + \frac{\sum fdm}{\sum f}$

$= 150 + \frac{38{,}200}{120}$

$= 150 + 318.33 = 468.33$

$\bar{x} = 468.33$

⇒

3. **STEP DEVIATION METHOD:** Step Deviation method is the most frequently used method of finding Arithmetic Mean in case of continuous series. This method is normally used when the class interval of the various classes is same. This method make the process of calculation simple. Following are the steps of this method:
   1. Calculate the Mid Values of the series represented by ‘m’.
   2. Take any value as assumed mean represented by ‘A’.
   3. Find out deviations of items from assumed mean. For that deduct Assumed value from mid values of the data. These deviations are represents as ‘dm’.
   4. Find out if all the values are divisible by some common factor ‘C’ and divide all the deviations with such common factor to find out dm’ which is dm/c
   5. Multiply the values of dm’ with corresponding frequency to find out product denoted by fdm’
   6. Find sum of the product so calculated represented by ∑fdm’.
   7. Find out the number of items in the series denoted by n.
   8. Calculate arithmetic mean dividing sum deviations of the observation with the number of observations using following formula:

$$\bar{x} = A + \frac{\sum fdm'}{\sum f} \times C$$

Where $\bar{x} = Mean$

A = Assumed Mean
N = Number of items
C = Common Factor
$\sum$fdm' = Sum of product of deviation after dividing with common factors and multiplying it with frequency.

Example 7. ***Use step deviation method to find*** $\overline{X}$ ***for the data given below*** **:**

Income $(Rs.)$	$1000 - 2000$	$2000 - 3000$	$3000 - 4000$	$4000 - 5000$	$5000 - 6000$	$6000 - 7000$
No. of Persons	4	7	16	20	15	8

Solution : Let the assumed mean $A = 4500$

Income $(Rs.)$ $C.I.$	No of Persons f	Mid Value m	$dm = m - A = (m - 4500)$	$dm' = \frac{dm}{C}$ $C = 1000$	fdm'
$1000 - 2000$	4	1500	-3000	-3	-12
$2000 - 3000$	7	2500	-2000	-2	-14
$3000 - 4000$	16	3500	-1000	-1	-16
$4000 - 5000$	20	4500	0	0	0
$5000 - 6000$	15	5500	1000	1	15
$6000 - 7000$	8	6500	2000	2	16
	$\sum f = 70$				$\sum fdm' = -11$

As $\overline{X} = A + \frac{\sum fdm'}{\sum f} \times C$

$$\overline{X} = 4500 + \frac{(-11)}{70} \times 1000$$
$$= 4500 - \frac{1100}{7}$$
$$= 4500 - 157.14$$
$$= 4342.86$$
$$\overline{X} = 4342.86$$

∴

OTHER SPECIAL CASE OF CONTINUOUS SERIES

7.6.4 ARITHMETIC MEAN IN CASE OF CUMULATIVE FREQUENCY SERIES:

The normal continuous series give frequency of the particular class. However, in case of cumulative frequency series, it does not give frequency of particular class rather it gives the total of frequency including the frequency of preceding classes. Cumulative frequency series may be of two types, that are 'less than' type and 'more than' type. For calculating Arithmetic mean in cumulative frequency series, we convert such series into the normal frequency series and then apply the same method as in case of normal series.

LESS THAN CUMULATIVE FREQUENCY DISTRIBUTION

Example 8. ***Find the mean for the following frequency distribution :***

Marks Less Than	10	20	30	40	50	60
No. of Students	5	15	40	70	90	100

Solution : Convert the given data into exclusive series :

Marks $C.I.$	No. of Students f	Mid Value m	$dm = m - A$ $A = 25$	$dm' = \frac{dm}{C}$ $C = 10$	fdm'
$0 - 10$	5	5	-20	-2	-10
$10 - 20$	$15 - 5 = 10$	15	-10	-1	-10
$20 - 30$	$40 - 15 = 25$	25	0	0	0
$30 - 40$	$70 - 40 = 30$	35	10	1	30
$40 - 50$	$90 - 70 = 20$	45	20	2	40
$50 - 60$	$100 - 90 = 10$	55	30	3	30
	$\sum f = 100$				$\sum fdm' = 80$

As $\bar{x} = A + \frac{\sum fdm'}{\sum f} \times C$

$\bar{x} = 25 + \frac{80}{100} \times 10 = 33$

$\bar{x} = 33$

⇒

⇒

MORE THAN CUMULATIVE FREQUENCY DISTRIBUTION

Example 9. ***Find the mean for the following frequency distribution***

Marks More Than	0	10	20	30	40	50	60	70	80	90
No. of Students	80	77	72	65	55	43	28	16	10	8

Solution : Convert the given data into exclusive series

Marks $C.I.$	No. of Students f	Mid Value	$dm = m - A$ $A = 55$	$dm' = \frac{dm}{C}$ $C = 10$	fdm'
$0 - 10$	$80 - 77 = 3$	5	-50	-5	-15
$10 - 20$	$77 - 72 = 5$	15	-40	-4	-20
$20 - 30$	$72 - 65 = 7$	25	-30	-3	-21
$30 - 40$	$65 - 55 = 10$	35	-20	-2	-20
$40 - 50$	$55 - 43 = 12$	45	-10	-1	-12
$50 - 60$	$43 - 28 = 15$	55	0	0	0
$60 - 70$	$28 - 16 = 12$	65	10	1	12
$70 - 80$	$16 - 10 = 6$	75	20	2	12
$80 - 90$	$10 - 8 = 2$	85	30	3	6
$90 - 100$	8	95	40	4	32
	$\sum f = 80$				$\sum fdm' = -26$

As $\bar{x} = A + \frac{\sum fdm'}{\sum f} \times C$

$$\bar{x} = 55 + \frac{(-26)}{80} \times 10$$

$$= 55 - \frac{13}{4}$$

$$= \frac{220 - 13}{4}$$

$$= \frac{207}{4} = 51.75$$

$$\bar{x} = 51.75$$

∴

⇒

7.6.5 ARITHMETIC MEAN IN CASE OF UNEQUAL CLASS INTERVAL SERIES:

Sometime the class interval between two classes is not same, for example 10-20, 20-40 etc. These series are known as unequal class interval series. However, it does not affect the finding of arithmetic mean as there is not precondition of equal class interval in case of arithmetic mean. So, mean will be calculated in usual manner.

Example 10. ***Calculate*** $\bar{x}$ ***if the data is given below*** **:**

$C.$.	$4 - 8$	$8 - 20$	$20 - 28$	$28 - 44$	$44 - 68$	$68 - 80$
f	3	8	12	21	10	6

Solution :

$C.I.$	f	Mid Value m	$dm = m - A$ $A = 26$	fdm
$4 - 8$	3	6	-20	-60
$8 - 20$	8	14	-12	-96
$20 - 28$	12	24	-2	-24
$28 - 44$	21	36	$+10$	210
$44 - 68$	10	56	$+30$	300
$68 - 80$	6	74	$+48$	288
	$\sum f = 60$			$\sum fdm = 618$

As $\overline{X} = A + \frac{\sum fdm}{\sum f}$

$\overline{X} = 26 + \frac{618}{60}$

$= 26 + 10.3 = 36.3$

$\overline{X} = 36.3$

$\Rightarrow$

$\Rightarrow$

7.6.6 COMBINED ARITHMETIC MEAN:

Sometime we have the knowledge of mean of two or more series separately but we are interested in finding the mean that will be obtained by taking all these series as one series, such mean is called combined mean. It can be calculated using the following formula.

$$\overline{X_{12}} = \frac{N_1\overline{X_1} + N_2\overline{X_2}}{N_1 + N_2}$$

Where $N_1 =$ Number of items in first series

$N_2 =$ Number of of items in second series

$\overline{X_1} =$ Mean of first series

and $\overline{X_2} =$ Mean of second series

Example 11. ***Find the combined mean for the following data***

	***Firm** A*	***Firm** B*
No. of Wage Workers	586	648
Average Monthly Wage $(Rs.)$	52.5	47.5

Solution : Combined mean wage of all the workers in the two firms will be

$$\overline{X_{12}} = \frac{N_1\overline{X_1} + N_2\overline{X_2}}{N_1 + N_2}$$

Where $N_1 =$ Number of workers in Firm A

$N_2 =$ Number of workers in Firm B

$\overline{X_1} =$ Mean wage of workers in Firm A

and $\overline{X_2} =$ Mean wage of workers in Firm B

We are given that

$N_1 = 586 \qquad N_2 = 648$

$\overline{X_1} = 52.5 \qquad \overline{X_2} = 47.5$

∴ Combined Mean, $\overline{X_{12}}$

$= \frac{(586 \times 52.5) + (648 \times 47.5)}{586 + 648}$

$= \frac{61,545}{1234}$

$= Rs. 49.9$

7.6.7 CORRECTING INCORRECT MEAN

Many a time it happens that we take some wrong items in the data or overlook some item. This results in wrong calculation of Mean. Later we find the correct values and we want to find out correct mean. This can be done using the following steps:

1. Multiply the incorrect mean of the data (incorrect $\overline{X}$) with number of items to find out incorrect $\sum \overline{X}$.
2. Now subtract all the wrong observation from the above values and add the correct observation to the above value to find out correct $\sum \overline{X}$.3. Now divide the correct $\sum \overline{X}$.with the number of observations to find correct mean.

Example12. ***Mean wage of*** 100 ***workers per day found to be*** 75**.** ***But later on it was found that the wages of two labourers*** $Rs. 98$ ***and*** $Rs. 69$ ***were misread as*** $Rs. 89$ ***and*** $Rs. 96$**.**

***Find out the correct mean wage*.**

Solution : We know that

Correct $\sum X =$ Incorrect $\sum X -$ (Incorrect items) $+$ (Correct Items)

Also $\overline{X} = \frac{\sum K}{N}$

⇒ Incorrect $\sum X = 100 \times 75 = 7500$

∴ Correct $\sum X = 7500 - (89 + 96) + (98 + 69)$

$= 7482$

⇒ Correct $\overline{X} = \frac{\text{Correct} \sum K}{N}$

$= \frac{7482}{100}$

$= 74.82$

DETERMINATION OF MISSING FREQUENCY

Example 13. ***Find the missing frequencies of the following series, if*** $\overline{X} = 33$ ***and*** $N = 100$

X	5	15	25	35	45	55
f	5	10	**?**	30	**?**	10

Solution : Let the missing frequencies corresponding to $X = 25$ and $X = 45$ be 'f_1' and 'f_2' respectively.

X	f	fX
5	5	25
15	10	150
25	f_1	$25f_1$
35	30	1050
45	f_2	$45f_2$
55	10	550
	$\sum f = 55 + f_1 + f_2$	$\sum fX = 1775 + 25f_1 + 45f_2$

Now, $N = 100$ (Given)

$$55 + f_1 + f_2 = 100$$

$$f_1 + f_2 = 45$$

$$\overline{X} = \frac{\sum fK}{N}$$

∴

⇒ …(i)

Also

$$\Rightarrow \quad 33 = \frac{1775+25f_1+45f_2}{100}$$

$$3300 = 1775 + 25f_1 + 45f_2$$

⇒

$$\Rightarrow \quad 25f_1 + 45f_2 = 1525 \quad \ldots\text{(ii)}$$

Solving (i) and (ii), we get

$$25 \times (f_1 + f_2 = 45) \qquad \Rightarrow 25f_1 + 25f_2 = 1125$$

$$1 \times (25f_1 + 45f_2 = 1525) \qquad \Rightarrow 25f_1 + 45f_2 = 1525$$

$$(-)(-) \qquad (-)$$

$$-20f_2 = -400$$

$$f_2 = \frac{400}{20} = 20$$

$\therefore$ $f_2 = 20$

Put $f_2 = 20$ in (i) f_1 $+ 20 = 45$

$\Rightarrow$ $f_1 = 45 - 20 = 25$

$\therefore$ $f_1 = 25$

$\therefore$ $f_1 = 25, f_2 = 20$

7.6.8 PROPERTIES OF ARITHMETIC MEAN

1. If we take the deviations of the observations from its Arithmetic mean and then sum up such deviations, then sum of such deviations will always be zero.
2. If we take the square of the deviations of items from its Arithmetic mean and then sum up such squares, the value obtained will always be less than the square of deviation taken from any other values.
3. If we have separate mean of two series, we can find the combined mean of the series.
4. If the value of all items in that data is increased or decreased by some constant value say 'k', then the Arithmetic mean is also increased or decreased by same 'k'. In other words if k is added to the items then actual mean will be calculated by deducting that k from the mean calculated.
5. If value of all items in the series is divided or multiplied by some constant 'k' then the mean is also multiplied or divided by the same constant 'k'. In other words if we multiply all observations by 'k' then actual mean can be calculated by dividing the mean to obtained by the constant 'k'.

7.6.9 MERITS OF ARITHMETIC MEAN

1. Arithmetic mean is very simple to calculate and it is also easy to understand.
2. It is most popular method of calculating the average.
3. Arithmetic mean is rigidly defined means it has a particular formula for calculating the mean.
4. Arithmetic mean is comparatively less affected by fluctuation in the sample.
5. It is most useful average for making comparison.
6. We can perform further treatment on Arithmetic mean.
7. We need not to have grouping of items for calculating Arithmetic mean. 8. Arithmetic mean is based on all the values of the data.

7.6.10 LIMITATIONS OF ARITHMETIC MEAN

1. The biggest limitation of Arithmetic mean is that it is being affected by extreme values.
2. If we have open end series, it is difficult to measure Arithmetic mean.
3. In case of qualitative data it is not possible to calculate Arithmetic mean.
4. Sometime it give absurd result like we say that there are 20 students in one class and 23 students in other class then average number of students in a class is 21.5, which is not possible because student cannot be in fraction.
5. It gives more importance to large value items than small value items.
6. Mean cannot be calculated with the help of a graph. 7. It cannot be located by just inspections of the items.

7.7 TEST YOUR UNDERSTANDING (A)

1. Following data pertains to the monthly salaries in rupees of the employees of a Mohanta Enterprises . Calculate the average salary per employ

   , 4100 , 4700 , 5400 , 2300 , 3400, 3700, 5100 , 5300 , 4700

2. Calculate mean for the following data using the shortcut method . 700 , 650 , 550, 750, 800 , 850, 650, 700 , 950

3. Following is the height of students of class tenth of a school. Find out the mean height of the students .

Height in Inches	64	65	66	67	68	69	70	71	72	73
No.of students	1	6	10	22	21	17	14	5	3	1

4. Calculate A.M for the following frequency distribution of Marks .

Marks	5	10	15	20	25	30	35	40
No of students	5	7	9	10	8	6	5	2

5. Calculate mean for the following data

Marks	5-15	15-25	25-35	35-45	45-55	55-65
No of Students	8	12	6	14	7	3

6. Calculate mean for the given data by step deviation method

C.I	0-10	10-20	20-30	30-40	40-50	50-60	60-70
f	8	12	14	16	15	9	6

7. From the following data , find the average sale per shop .

Sales in '000; units	10-12	13-15	16-18	19-21	22-24	25-27	28-30
No. shops	34	50	85	60	30	15	7

8. For the following data (Cumulative Series) , find the average income .

Income Below in (Rs.)	30	40	50	60	70	80	90
No. of persons	16	36	61	76	87	95	100

9. Calculate the average marks for the following cumulative frequency distribution .

Marks Above	0	10	20	30	40	50	60	70	80	90
No of students	80	77	72	65	55	43	28	16	10	8

10. For a group of 50 male workers, their average monthly wage Rs.6300 and for a group of 40 female workers this average is Rs.5400 . Find the average monthly wage for the combined group of all the workers.
11. The average marks of 100 students is given to be 45 . But later on it was found that the marks of students getting 64 was misread as 46 . Find the correct mean.

12. Find missing frequency when mean is 35 and number is 68.

X; 0-10 10-20 20-30 30-40 40-50 50-60

F: 4 10 12 ? 20 ?

13. The mean age of combined group of men and women is 30 years. The mean age of group of men is 32 years and women is 27 years. Find the percentage of men and women in the group

Answers

1) 4170
2) 733.30
3) 68.13 inches
4) 20.48
5) 31.8
6) 33.625
7) 17.8 (in 000 units)
8) 48
9) 51.75
10) 5900
11) 45.18
12) 10,12
13) Men 60%

7.8 GEOMETRIC MEAN:

Some time we deal with such quantities or items that change over a period of time. In that case we are interested in finding the rate of change in the item over the period of time. In other words we can say that we are interested in finding the rate of growth or rate of decline in the item. For example we want to know average rate of growth in the population, growth in national income of the country or annual decline rate in the value of machinery etc. In that case the most appropriate measure of average is geometric mean. The geometric mean is represented by G.M. we can define Geometric mean as:

" Geometric mean of N items is root nth of the product of item" Symbolically we can write Geometric mean as:

$$G.M. = \sqrt[N]{X_1 \times X_2 \times X_3 \times \ldots\ldots\ldots\ldots X_N}$$

Where $G.M. = Geometric\ Mean$

N = Number of items

X_1, X_2, X_3 .... = Various items or observations.

7.8 .1 GEOMETRIC MEAN IN INDIVIDUAL SERIES

Following are the steps for calculating Geometric mean in the individual series

1. Take the logarithm of all the values.
2. Find the sum of the values after taking the logarithm.
3. Divide the sum with number of items.
4. Find out antilogarithm of the resultant figure.

$$\text{Geometric Mean or } G.M. = \text{Antilog}\left(\frac{\sum \log X}{N}\right)$$

Example 14. ***Calculation geometric mean for the data given below***

7.7 CHECK YOUR UNDERSTANDING (A)					

Solution :

X	log X	
60	log 60	1.7782
75	log 75	1.8751
90	log 90	1.9542
90	log 90	1.9542
90	log 90	1.9542
$N = 5$		$\sum \log X = 9.5159$

As Geometric Mean, $G = \text{Antilog}\left(\frac{\sum \log K}{N}\right)$

$G = \text{Antilog}\left(\frac{9.5159}{5}\right)$

$= \text{Antilog}(1.9032) = 80$

⇒ $G = 80$

7.8.2 GEOMETRIC MEAN IN DISCRETE SERIES

Following are the steps for calculating Geometric mean in the individual series

1. Take the logarithm of all the values.
2. Multiply the logarithm with the corresponding frequency of the items.
3. Find the sum of the prodcut.
4. Divide the sum with number of items.
5. Find out antilogarithm of the resultant figure.

$$\text{Geometric Mean or } G.M = \text{Antilog}\left(\frac{\sum f \log X}{N}\right)$$

Example 15. ***Find G, the geometric mean, for the following data***

X	15	25	35	45	55
f	5	10	15	7	4

Solution :

X	f	log X		f log X
15	5	log 15	1.1761	5.8805
25	10	log 25	1.3979	13.9790
35	15	log 35	1.5441	23.1615
45	7	log 45	1.6532	11.5724
55	4	log 55	1.7404	6.9616
	$\sum f = N = 41$			$\sum f \log X = 61.5550$

As Geometric Mean, $G = \text{Antilog}\left(\frac{\sum f \log K}{N}\right)$

∴ $G = \text{Antilog}\left(\frac{61.5550}{41}\right)$

$= \text{Antilog}(1.5013) = 31.72$

$G = 31.72$

7.8.3 GEOMETRIC MEAN IN DISCRETE SERIES

Finding G.M. in the continuous series is same as in case of discrete series except that we have to find the mid values of the class intervals. Rest all the steps are same. Following are the steps for calculating Geometric mean in the individual series

1. Find mid values of each class interval.
2. Take the logarithm of all the mid values.
3. Multiply the logarithm with the corresponding frequency of the items.
4. Find the sum of the product.
5. Divide the sum with number of items.
6. Find out antilogarithm of the resultant figure.

$$\text{Geometric Mean or } G.M = \text{Antilog}\left(\frac{\sum f \log X}{N}\right)$$

Example 16. ***Find*** GM

$C.I.$	$1.5-2.5$	$2.5-3.5$	$3.5-4.5$	$4.5-5.5$	$5.5-6.5$
f	10	15	7	18	12

Solution :

$C.I.$	f	Mid Value x	$\log x$	$f \log x$
$1.5-2.5$	10	2	0.3010	3.0100
$2.5-3.5$	15	3	0.4771	7.1565
$3.5-4.5$	7	4	0.6021	4.2147
$4.5-5.5$	18	5	0.6990	12.5820
$5.5-6.5$	12	6	0.7782	9.3384
	$\sum f = N = 62$			$\sum f \log x = 36.3016$

As $G = \text{Antilog}\left(\frac{\sum f \log K}{\sum f}\right)$

$G = \text{Antilog}\left(\frac{36.3016}{62}\right)$

$= \text{Antilog}(0.5855) = 3.850$

$G = 3.850$

7.8.4 MERITS OF GEOMETRIC MEAN.

1. Geometric mean is rigidly defined.
2. It is very suitable for calculating growth or decline rate.
3. Its calculation is based on all the items under observation.
4. Further mathematical treatment can be applied to it.
5. Like Arithmetic mean shows biasness for higher values, Geometric mean shows business for lower values which is useful in many situations like price analysis.
6. It is comparatively less affected by Extreme value. 7. It does not change much with the change in sample.

7.8.5 DEMERITS OF GEOMETRIC MEAN

1. It is comparatively difficult to calculate.
2. It is also difficult to understand and interpret .
3. It cannot be calculated if negative values are present in the series.
4. Even if a single observation is zero in the series the geometric mean becomes zero.

7.9 HARMONIC MEAN

Harmonic mean is a average that is used for finding average rate like we are interested in finding the average speed of the vehicle or we know that three persons take 10, 12 and 14 hours to complete a work individually and we are interested in finding average time. In this case there is reciprocal relation between the time taken and speed of the work, more is the time taken by the person less is the speed and less is the time taken by the person more is the speed. In these situations we can use harmonic mean. In harmonic mean we give more weightage to smaller items and less weightage to larg items. it is most useful measure of Central tendency for calculating the ratio. Harmonic mean can be defined as

"It is reciprocal of Arithmetic mean of reciprocal of the observations." Mathematically we can write Harmonic Mean as

$$Harmonic\ Mean\ or\ H.M. = \frac{N}{\Sigma\left(\frac{1}{X}\right)}$$

Where $H.M. = Harmonic\ Mean$

N = Number of items ∑X

= Sum of observation.

7.9.1 HARMONIC MEAN IN INDIVIDUAL SERIES

Following are the steps for calculating Geometric mean in the individual series

1. Take the reciprocal of all the values.
2. Find the sum of the reciprocal of the values.
3. Find the arithmetic mean of sum of reciprocal.

4. Reciprocal to the arithmetic mean so calculated is Harmonic Mean to the data.

$$Harmonic\ Mean\ or\ H.M. = \frac{N}{\sum(\frac{1}{X})}$$

Example 17. ***Find the*** $H.M.$ ***for the data given below :***

X	35	45	89	87	66	76	110	135

Solution :

X	$\frac{1}{X}$
35	0.0286
45	0.0222
89	0.0112
87	0.0115
66	0.0151
76	0.0131
110	0.0091
135	0.0074
$N = 8$	$\sum(\frac{1}{X}) = 0.1184$

As Harmonic Mean, $H.M. = \frac{N}{\sum(\frac{1}{X})}$

$$= \frac{8}{0.1184} = 67.57$$

∴ $H.M. = 67.57$

7.9.2 HARMONIC MEAN IN DISCRETE SERIES

Following are the steps for calculating Geometric mean in the individual series

1. Take the reciprocal of all the values.
2. Multiply the reciprocal with corresponding frequencies to find product.
3. Find the sum of the product of reciprocals and frequencies.
4. Find the arithmetic mean of sum of reciprocal.
5. Reciprocal to the arithmetic mean so calculated is Harmonic Mean to the data.

$$Harmonic\ Mean\ or\ H.M. = \frac{N}{\sum (f \times \frac{1}{X})}$$

Example 18. ***Find*** $H.M.$

X	20	50	55	65
f	10	20	15	10

Solution :

X	f	1/X	$\frac{f}{X}$
20	10	1/20	$10/20 = 0.5$
50	20	1/50	$20/50 = 0.4$
55	15	1/55	$15/55 = 0.2727$
65	15	1/65	$15/65 = 0.2308$
	$\sum f = N = 60$		$\sum (\frac{f}{X}) = 1.4035$

As Harmonic Mean, $H.M. = \frac{N}{\sum(\frac{1}{X})}$

$$= \frac{60}{1.4035} = 42.75$$

∴ $H.M. = 42.75$

7.9.3 HARMONIC MEAN IN CONTINUOUS SERIES

Calculation of Harmonic mean in continuous and discrete series is almost same except that in continuous series we take mean value of the class intervals. Following are the steps for calculating Geometric mean in the individual series

1. Find mid value of each class.
2. Take the reciprocal of all the mid values.
3. Multiply the reciprocal with corresponding frequencies to find product.
4. Find the sum of the product of reciprocals and frequencies.
5. Find the arithmetic mean of sum of reciprocal.
6. Reciprocal to the arithmetic mean so calculated is Harmonic Mean to the data.

$$Harmonic\ Mean\ or\ H.M. = \frac{N}{\sum (f \times \frac{1}{X})}$$

Example 19. ***Find Harmonic mean, if the data is given as***

$C.I.$	$0-100$	$100-200$	$200-300$	$300-400$	$400-500$
f	2	7	13	5	3

Solution :

$C.I.$	f	Mid Value x	$\frac{f}{x}$
$0-100$	2	50	0.040
$100-200$	7	150	0.0467
$200-300$	13	250	0.052
$300-400$	5	350	0.0143
$400-500$	3	450	0.0067
	$\sum f = N = 30$		$\sum (\frac{f}{x}) = 0.1597$

As Harmonic Mean, $H.M. = \frac{N}{\Sigma(\frac{f}{x})}$

$$= \frac{30}{0.1597} = 180.79$$

∴ $H.M. = 180.79$

7.9.4 MERITS OF HARMONIC MEAN

1. Harmonic mean is rigidly defined.
2. Its calculation is based on all observations.
3. Further algebraic treatment can be applied on it.
4. It is not affected by fluctuation in the sampling.
5. In the problem related to time and work, time and speed etc, this is the best average to measure central tendency.

7.9.5 LIMITATIONS OF HARMONIC MEAN

1. This is least understood average.
2. Calculation of reciprocal values is not easy task.
3. More weightage is given to small items and big items get lesser weightage. 4. If observation has zero or negative value , it cannot be calculated.

7.10 TEST YOUR UNDERSTANDING (B)

1. Find Geometric Mean

X	10	110	35	120	50	59	60	7

2. It arithmetic mean of data is 12.5 and G.M. is 10 find the difference between the items.

3. Calculate G.M.

X	10	15	18	25
f	*2*	*3*	*5*	*4*

4. Find G.M from the following data

X	*3834*	*382*	*63*	*9*	*.4*	*.009*	*.0005*

5. Find the G.M of 2, 4 and 8 and prove it is less than A.M

6. Find G.M

C. I.	$0-10$	$10-20$	$20-30$	$30-40$	$40-50$
f	4	8	10	6	7

7. Find H.M.

X	10	20	40	60	120

8. Calculate H.M.

X	10	20	25	40	50 5
f	20	30	50	15	

9. Find H.M from the following data

X	*3834*	*382*	*63*	*.8*	*.4*	*.03*	*.009*	*.0005*

10. Find H.M

C. I.	$10-20$	$20-30$	$30-40$	$40-50$	$50-60$

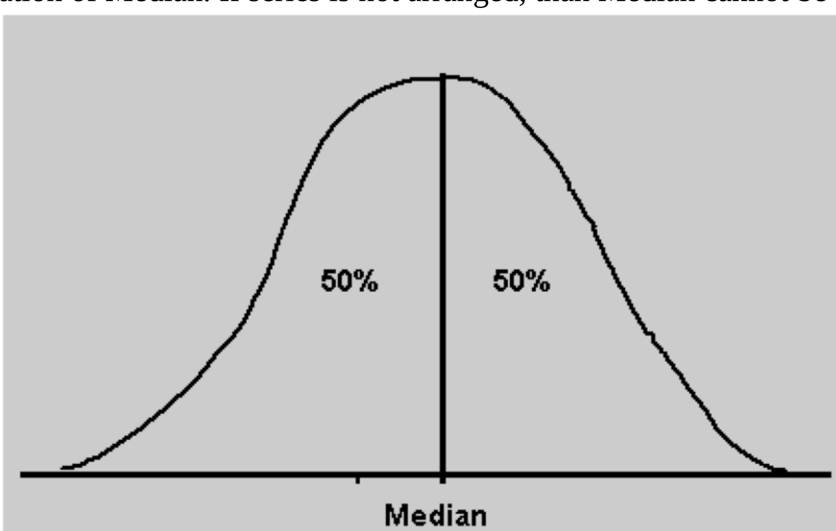
f	4	6	10	7	3

Answers

1) 46.56
2) 15
3) 18.2
4) 1.609
6) 22.06
7) 25
8) 20.08
9) 0.00373
10) 29.88

7.11 MEDIAN

Median is the positional measure of Central tendency. It means the median does not depend upon the value of the item under the observation, rather it depend on the position of the item in the series. Median is a value that divide the series exactly in two equal parts, it means 50% of the observation lies below the median and 50% of the observations lies above the median. However, it is important to arrange the series either in ascending order or in descending order before calculation of Median. If series is not arranged, than Median cannot be calculated



For calculating Median

1. **Series should be in ascending or descending order.**
2. **Series should be exclusive, not inclusive.**

7.11.1 MEDIAN IN CASE OF INDIVIDUAL SERIES.

For calculating the median in individual series, following are the steps:

1. Arrange the series in ascending or descending order.
2. Calculate the number of observations. It is denoted by N.
3. Calculate the $\left(\frac{N+1}{2}\right)^{th}$ term
4. Corresponding value to this item is the median of the data
5. In case there are even number of items in the series, this value will be in fraction. In that case take the arithmetic mean of the adjacent items in which Median is falling. For example if it is 4.5 than take arithmetic mean of 4^{th} item and 5^{th} item.

Median = value of $\left(\frac{N+1}{2}\right)^{th}$ **term**

When the number of observations N is odd

Example 20. ***Calculation median from the following observations*** **:**

$15, 17, 19, 22, 18, 47, 25, 35, 21$

Solution : Arranging the given items in ascending order, we get

$15, 17, 18, 19, 21, 22, 25, 35, 47$

Now Median, $M = \text{Size of } \left(\frac{N+1}{2}\right)^{th} \text{ item}$

$M = \text{Size of } \left(\frac{9+1}{2}\right)^{th} \text{item}$

$= \text{Size of } 5^{th} \text{ item}$

$= 21$

$\Rightarrow \quad = 21$

When the number of observation N is even

Example 21. ***Find median from the following data***

$28, 26, 24, 21, 23, 20, 19, 30$

Solution : Arranging the given figures in ascending order, we get

$20, 21, 23, 24, 26, 28, 30 19,$

Now Median, $M = \text{Size of } \left(\frac{N+1}{2}\right)^{th} \text{item}$

$$M = \text{Size of } \left(\frac{8+1}{2}\right)^{th} \text{item}$$

$$= \text{Size of } 4.5^{th} \text{ item}$$

$$= \frac{4^{th}\text{item} + 5^{th}\text{item}}{2}$$

$$= \frac{23+24}{2} = \frac{47}{2} = 23.5$$

$\Rightarrow \quad M = 23.5$

7.11.2 MEDIAN IN CASE OF DISCRETE SERIES

Following are the steps in case of discrete series:

1. Arrange the data in ascending or descending order.
2. Find the cumulative frequency of the series.
3. Find the $\left(\frac{N+1}{2}\right)^{th}$ term
4. Now look at this term in the cumulative frequency of the series.
5. Value against which such cumulative frequency falls is the median value.

Median = value of $\left(\frac{N+1}{2}\right)^{th}$ term

Example 22. ***Calculate the value of median, if the data is as given below*** **:**

Height (in cms.)	110	125	250	200	150	180
No. of Students	8	12	3	10	13	15

Solution : Arranging the given data in ascending order, we get

Height (in cms.)	No. of Students f	Cumulative Frequency $C \cdot f$
110	8	$8\ (1-8)$
125	12	$20\ (9-20)$
150	13	$33\ (21-33)$
180	15	$48\ (34-48)$
200	10	$58\ (49-58)$
250	3	$61\ (59-61)$
	$\sum f = N = 61$	

Now Median, $M = \text{Size of } \left(\frac{N+1}{2}\right)^{th}$ item

$M = \text{Size of } \left(\frac{6+1}{2}\right)^{th}$ item

$= \text{Size of } 31^{st}$ item

$= 150 \Rightarrow$

Median, $M = 150$ cms.

7.11.3 MEDIAN IN CASE OF CONTINUOUS SERIES

Following are the steps in case of continuous series:

1. Arrange the data in ascending or descending order.
2. Find the cumulative frequency of the series.
3. Find the $\left(\frac{N}{2}\right)^{th}$ term
4. Now look at this term in the cumulative frequency of the series. The value equal to or higher than term calculated in third step is the median class.
5. Find median using following formula.

$$M = L + \frac{\frac{N}{2} - C\cdot f}{f} \times \text{i}$$

Where M = Median

L = Lower Limit of Median Class
N = Number of Observations.

c.f. = Cumulative frequency of the Median Class. f
= Frequency of the class preceeding Median Class. i
= Class interval of Median Class

Example 23. ***Calculate Median***

Marks	$5-10$	$10-15$	$15-20$	$20-25$	$25-30$	$30-35$
No. of Students	8	7	14	16	9	6

Solution :

$C.I.$	No. of Students f	Cumulative Frequency $C\cdot f$
$5-10$	8	$8\ (1-8)$
$10-15$	7	$15\ (9-15)$
$15-20$	14	$29\ (16-29)$

$20-25$	16	$45\ (30-45)$
$25-30$	9	$54\ (46-54)$
$30-35$	6	$60\ (55-60)$
	$\sum f = N = 60$	

Median, $M =$ Size of $\left(\frac{N}{2}\right)^{th}$ item

$M =$ Size of $\left(\frac{60}{2}\right)^{th}$ item

$=$ Size of 30^{th} item

$\Rightarrow$ Median lies in the class interval $20-25$

As Median, $M = L + \frac{\frac{N}{2} - C \cdot f}{f} \times i$

Here $L =$ Lower limit of the median class $= 20$

$N = 60$ $C \cdot f = 29$ $f = 16$ $i =$ Class – length of the median class $= 5$

$\therefore$ $M = 20 + \frac{(30-29)}{16} \times 5$

$= 20 + \frac{5}{16}$

$= 20 + 9.312 = 29.312$

$\Rightarrow$ $M = 29.312$

Inclusive Series – It must be converted to Exclusive Series before calculation of the Median.

Example 24. ***Find Median from the given data***

X	$10-19$	$20-29$	$30-39$	$40-49$	$50-59$	$60-69$	$70-79$	$80-89$
f	6	53	85	56	21	16	4	4

Solution : Converting the given data into exclusive form, we get

$$\left[\text{Correction factor} = \frac{L_2 - U_1}{2} = \frac{20-19}{2} = \frac{1}{2} = 0.5\right]$$

(0.5 is subtracted from all lower limits and added to all upper limits)

X	f	Cumulative frequency $C \cdot f$	
$9.5-19.5$	6	6	$(1-6)$
$19.5-29.5$	53	59	$(7-59)$
$29.5-39.5$	85	144	$(60-144)$
$39.5-49.5$	56	200	$(145-200)$
$49.5-59.5$	21	221	$(201-221)$
$59.5-69.5$	16	237	$(222-237)$
$69.5-79.5$	4	241	$(238-241)$
$79.5-89.5$	4	245	$(242-245)$
	$\sum f = N = 245$		

Median, $M =$ Size of $\left(\frac{N}{2}\right)^{th}$ item

$M =$ Size of $\left(\frac{245}{2}\right)^{th}$ item

$=$ Size of 122.5^{th} item

∴ The real class limits of the median class $= (29.5 - 39.5)$

So $M = L + \frac{\left(\frac{N}{2} - C \cdot f\right)}{f} \times i$

⇒ $M = 29.5 + \left(\frac{122.5-59}{85}\right) \times 10$

$= 29.5 + \left(\frac{63.5}{85} \times 10\right)$

$= 29.5 + \left(\frac{635}{85}\right)$

$= 29.5 + 7.47 = 36.97$

$M = 36.97$

⇒

Cumulative Series (More than and Less than)

Example 25. ***Find median, if the data is as given below*** **:**

Marks More than	20	35	50	65	80	95
No. of Students	100	94	74	30	4	1

Solution : Converting the given data into class – interval form, we get

Marks $C.I.$	Frequency f	Cumulative Frequency $C \cdot f$
$20 - 35$	$100 - 94 = 6$	$6 \quad (1 - 6)$
$35 - 50$	$94 - 74 = 20$	$26 \quad (7 - 26)$
$50 - 65$	$74 - 30 = 44$	$70 \quad (27 - 70)$
$65 - 80$	$30 - 4 = 26$	$96 \quad (71 - 96)$
$80 - 95$	$4 - 1 = 3$	$99 \quad (97 - 99)$
$95 - 110$	1	$100 \quad (100)$
	$\sum f = N = 100$	

Now Median, $M =$ Size of $\left(\frac{N}{2}\right)^{th}$ item

$$M = \text{Size of } \left(\frac{100}{2}\right)^{th} \text{item}$$

$$= \text{Size of } 50^{th} \text{ item}$$

$\Rightarrow$ Median lies in the class interval$= 50 - 65$

So $M = L + \frac{\left(\frac{N}{2} - C \cdot f\right)}{f} \times i$

$$M = 50 + \left(\frac{50 - 26}{44}\right) \times 15$$

$\Rightarrow$

$$= 50 + \left(\frac{24}{44} \times 15\right)$$

$$= 50 + 8.18 = 58.18$$

$\Rightarrow$ $M = 58.18$

Example 26. ***Find median, if the data is as given below*** **:**

Marks Less than	10	0	30	40	50	60	70	80
No. of Students	20	30	50	94	96	127	198	250

Solution : Converting the given data into class interval form, we get

Marks $C.I.$	No. of Students f	Cumulative Frequency 20 $C \cdot f$
$0 - 10$	20	$(1 - 20)$
$10 - 20$	$30 - 20 = 10$	$(21 - 30)$
$20 - 30$	$50 - 30 = 20$	$(31 - 50)$
$30 - 40$	$94 - 50 = 44$	$(51 - 94)$
$40 - 50$	$96 - 94 = 2$	$(95 - 96)$
$50 - 60$	$127 - 96 = 31$	$(97 - 127)$
$60 - 70$	$198 - 127 = 71$	$(128 - 198)$
$70 - 80$	$250 - 198 = 52$	$(199 - 250)$ 30 50 94 96 127 198 250
	$\sum f = N = 250$	

Now Median, $M =$ Size of $\left(\frac{N}{2}\right)^{th}$ item

$$M = \text{Size of } \left(\frac{250}{2}\right)^{th} \text{ item}$$

$$= \text{Size of } 125^{th} \text{ item}$$

$\Rightarrow$ Median lies is the class – interval$=$ $50 - 60$

So $M = L + \frac{\frac{N}{2} - C \cdot f}{f} \times i$

$M = 50 + (\frac{125-96}{31}) \times 10$

$\Rightarrow$

$= 50 + (\frac{29}{31} \times 10)$

$= 50 + \frac{290}{31}$

$= 50 + 9.35 = 59.35$

$\Rightarrow \quad M = 59.35$

Mid – Value Series

Example 27. ***Find the value of median for the following data*** **:**

Mid Value	15	25	35	45	55	65	75	85	95
f	8	26	45	72	116	60	38	22	13

Solution : It is clear form the mid – value that the class size is 10. For finding the limits of different classes, apply the formula :

$L = m - \frac{i}{2}$ and $U = m + \frac{i}{2}$

Where, L and U denote the lower and upper limits of different classes, 'm' denotes the mid – value of the corresponding class interval and 'i' denotes the difference between mid values. ∴ Corresponding to mid – value '15', we have $L = 15 - \frac{10}{2}$ and $U = 15 + \frac{10}{2}$

i.*e.* $C.I. = 10 - 20$

Similarly other class intervals can be located

Mid Value	f	$C.I.$	Cumulative Frequency $C \cdot f$
15	8	$10 - 20$	8 $(1 - 8)$
25	26	$20 - 30$	34 $(9 - 34)$
35	45	$30 - 40$	79 $(35 - 79)$
45	72	$40 - 50$	151 $(80 - 151)$

55	116	$50-60$	267 (152 − 267)
65	60	$60-70$	327 (268 − 327)
75	38	$70-80$	365 (328 − 365)
85	22	$80-90$	387 (366 − 387)
95	13	$90-100$	400 (388 − 400)
	$N=100$		

Now Median, $M = \text{Size of } \left(\frac{N}{2}\right)^{th}$ item

$M = \text{Size of } \left(\frac{400}{2}\right)^{th}$ item

$= \text{Size of } 200^{th}$ item

⇒ Median lies is the class – interval$=50-60$

So $M = L + \frac{\frac{N}{2} - C \cdot f}{f} \times i$

$M = 50 + \left(\frac{200-151}{116}\right) \times 10$

$= 50 + \left(\frac{49}{116} \times 10\right)$

$= 50 + \frac{490}{116}$

⇒

$= 50 + 4.224 = 54.224$

⇒ $M = 54.224$

Determination of Missing Frequency

Example 28. ***Find the missing frequency in the following distribution if*** $N = 72$, $Q_1 = 25$ ***and*** $Q_3 = 50$

$C.I.$	$0-10$	$10-20$	$20-30$	$30-40$	$40-50$	$50-60$	$60-70$	$70-80$
f	4	8	$-$	19	$-$	10	5	$-$

Solution : Let the missing frequencies be f_1, f_2 and f_3 respectively.

$C.I.$	f	Cumulative Frequency $C \cdot f$
$0-10$	4	4
$10-20$	8	12
$20-30$	f_1	$12+f_1$
$30-40$	19	$31+f_1$
$40-50$	f_2	$31+f_1+f_2$
$50-60$	10	$41+f_1+f_2$
$60-70$	5	$46+f_1+f_2$
$70-80$	f_3	$46+f_1+f_2+{}_3$
	$N=72=\sum f$ $\sum f = 46+f_1+f_2+f_3$	

Now

$N=72$

$=\sum f$

$=46+f_1+f_2+f_3$

$\Rightarrow \quad f_1+f_2+f_3 = 72-46 = 26$

$\Rightarrow \quad f_1+f_2+f_3 = 26$ …(i)

Also, $\quad Q_1 = 25$ (Given)

$\Rightarrow \quad Q_1$ lies in the class – interval $20-30$

$\Rightarrow \quad Q_1 = L + \dfrac{\frac{N}{4} - C \cdot f}{f} \times i$

$25 = 20 + \dfrac{\frac{72}{4} - 12}{f_1} \times 10$

$25 = 20 + \dfrac{18-12}{f_1} \times 10$

$25 - 20 = \dfrac{6}{f_1} \times 10$

$5f_1 = 60$

$f_1 = \dfrac{60}{5}$

$f_1 = 12$

$\Rightarrow$ …(ii)

Similarly, we are given that

$Q_3 = 50$

$\Rightarrow \quad Q_3$ lies in the class – interval $50 - 60$

$\Rightarrow \quad Q_3 = L + \dfrac{\frac{3N}{4} - C \cdot f}{f} \times i$

$50 = 50 + \dfrac{\frac{3 \times 72}{4} - (31 + f_1 + f_2)}{10} \times 10$

$50 = 50 + \dfrac{54 - (31 + 12 + f_2)}{1}$

($\because f_1 = 12$ By (ii))

$50 - 50 = 54 - (43 + f_2)$

$0 = 54 - (43 + f_2)$

$43 + f_2 = 54$ $f_2 =$

$54 - 43$

$\Rightarrow \quad f_2 = 11$ …(iii)

Putting (ii) and (iii) in (i), we get

$f_1 + f_2 + f_3 = 26$

$12 + 11 + f_3 = 26$

$23 + f_3 = 26$ $f_3 =$

$26 - 23$

$\Rightarrow \quad f_3 = 3$

7.11.4 MERITS OF MEDIAN

1. Median is easy to calculalte.
2. It is capable of Graphic presentation.
3. It is possible even in case of open ende series.
4. This is rigidly defined.
5. It is not affected by extreme values. **6.** In case of qualitative data, it is very useful**.**

7.11.5 LIMITATIONS OF MEDIAN

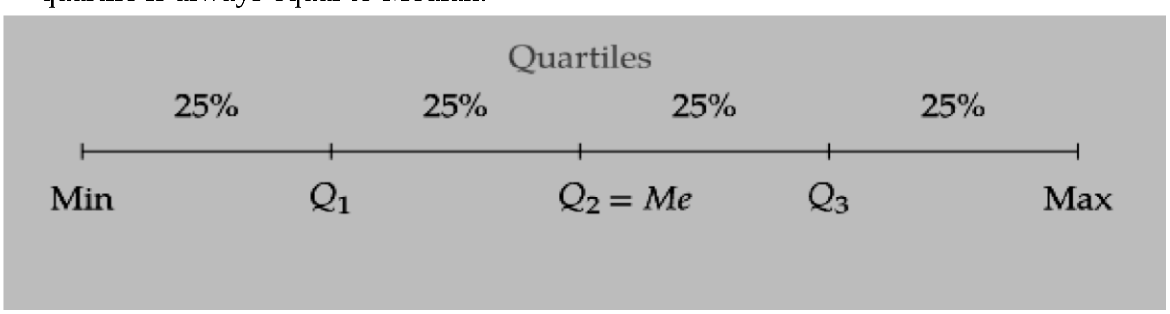
1. It is not capable of further algebraic treatment.
2. It is positional average and is not based on all observation.
3. It is very much affected by fluctuation in sampling.
4. Median needs arrangement of data before calculation.
5. In case of continuous series it assumes that values are equally distributed in a particular class.

7.12 OTHER POSITIONAL MEASURES (QUARTILES, DECILES AND PERCENTILES)

As median divide the series into two equal parts, there are many other positional measures also. These Positional measures are also known as partition values. Following are some of the positional measure

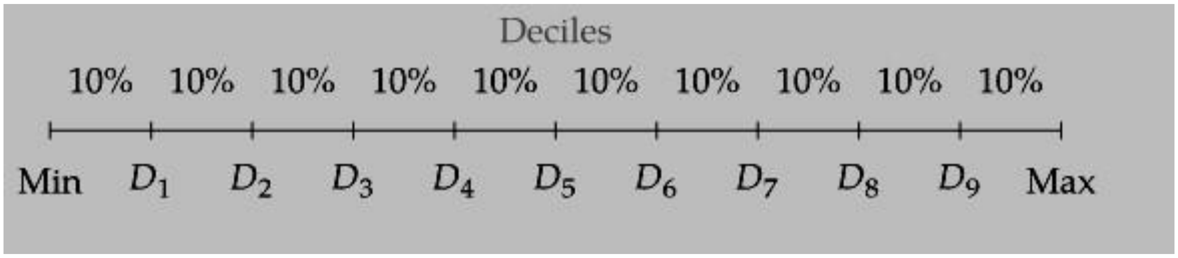
7.12.1 QUARTILES

Quartile are the values that divide the series in four equal parts. There are total three quarter in number denoted by Q1, Q2 and Q3. First quartile is placed at 25% of the items, second quartile at 50% of the items, third quartile at 75% of the items. The value of second quartile is always equal to Median.



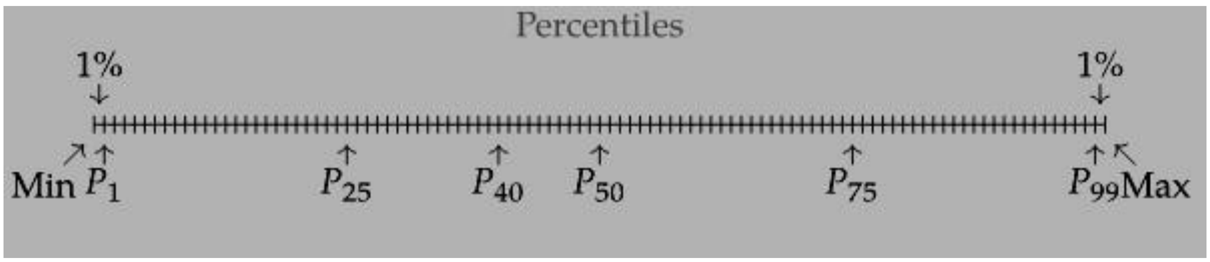
7.12.2 DECILES

Deciles are the values that divide the series in ten equal parts. There are total nine Deciles in number denoted by D1, D2, D3 and so on upto D9. The first decile is placed at 10% of the items, second quartile at 20% of the items, similarly last at 90% of the items. The value of fifth Decile is always equal to Median.



7.12.3 PERCENTILE

Percentiles are the values that divide the series in hundred equal parts. There are total ninety nine Percentiles in number denoted by P1, P2, P3 and so on upto P99. The first Percentile is placed at 1% of the items, second quartile at 2% of the items, similarly last at 99% of the items. The value of fifteeth Percentile is always equal to Median.



The methods of finding positional measures are same as in case of median. However following are the formulas that can be used for finding positional measures.

Partition Value	Individual Series	Discrete Series	Continuous Series	Continuous Series
Q1	Value of $(\frac{N+1}{4})^{th}$ item	Value of $(\frac{N+1}{4})^{th}$ item	Value of $(\frac{N}{4})^{th}$ item	$L + \frac{\frac{N}{4} - C \cdot f}{f} \times i$
Q3	Value of $3\,(\frac{N+1}{4})^{th}$ item	Value of $3\,(\frac{N+1}{4})^{th}$ item	Value of $3\,(\frac{N}{4})^{th}$ item	$L + \frac{3(\frac{N}{4}) - C \cdot f}{f} \times i$
D6	Value of $6\,(\frac{N+1}{10})^{th}$ item	Value of $6\,(\frac{N+1}{10})^{th}$ item	Value of $6\,(\frac{N}{10})^{th}$ item	$L + \frac{6(\frac{N}{10}) - C \cdot f}{f} \times i$

P40	Value of $40\left(\frac{N+1}{100}\right)^{th}$ item	Value of $40\left(\frac{N+1}{100}\right)^{th}$ item	Value of $40\left(\frac{N}{100}\right)^{th}$ item	$L + \frac{40\left(\frac{N}{100}\right) - C \cdot f}{f} \times i$

Similarly all the values can be calculated.

(A) INDIVIDUAL SERIES

Example 29. ***From the data given below, determine*** Q_1, Q_3, D_5, P_{40}**.**

Marks in Economics	18	20	25	24	32	50	55	45	55	40	60

Solution : Arranging the given figures in ascending order, we get

S. No.	1	2	3	4	5	6	7	8	9	10	11	$N = 11$
Marks	18	20	24	25	32	40	45	50	52	55	66	

Now $Q_1 = \text{Value of} \left(\frac{N+1}{4}\right)^{th}$ item

$= \text{Value of} \left(\frac{11+1}{4}\right)^{th}$ item

$= \text{Value of } 3^{rd} \text{ item} = 24$

$\therefore \quad Q_1 = 24$

$Q_3 = \text{Value of } 3\left(\frac{N+1}{4}\right)^{th}$ item

$= \text{Value of } 3\left(\frac{11+1}{4}\right)^{th}$ item

$= \text{Value of } 9^{th} \text{ item} = 52$

$\therefore \quad Q_3 = 52$

$D_5 = \text{Value of } 5\left(\frac{N+1}{10}\right)^{th}$ item

$= \text{Value of } 5\left(\frac{11+1}{10}\right)$item

$= \text{Value of } 6^{th} \text{ item} = 40$

$\therefore \quad D_5 = 40$

$40\left(\frac{N+1}{100}\right)^{th}$

th

$P_{40} =$ Value of item

$=$ Value of $40\left(\frac{12}{100}\right)$item

$=$ Value of 4.8^{th} item $= 24$

$\Rightarrow$ $P_{40} = 4^{th}\text{item} + 0.8\,(5^{th}\text{item} - 4^{th}\text{item})$

$= 25 + 0.8\,(32 - 25)$

$= 25 + 0.8\,(7)$

$= 25 + 5.6 = 30.6$

$\therefore$ $P_{40} = 30.6$

(B) DISCRETE SERIES

Example 30. ***Form the following data, compute*** O_1, O_3, D_8 ***and*** P_{70}**.**

X	110	120	130	140	150	160	170
f	2	3	5	10	5	3	2

Solution :

X	f	$C.I.$
110	2	2 $(1 - 2)$
120	3	5 $(3 - 5)$
130	5	10 $(6 - 10)$
140	10	20 $(11 - 20)$
150	5	25 $(21 - 25)$
160	3	28 $(26 - 28)$
170	2	30 $(29 - 30)$
	$\sum f = N = 30$	

Now $Q_1 =$ Value of$\left(\frac{N+1}{4}\right)^{th}$ item

$=$ Value of$\left(\frac{30+1}{4}\right)^{th}$item

$= \text{Value of } 23.25^{th} \text{ item} = 150$

$\therefore \quad Q_1 = 150$

th item

$Q_3 \quad = \text{Value of } 3\left(\frac{N+1}{4}\right)$

$3\left(\frac{30+1}{4}\right)^{th}$

$= \text{Value of}$item

$= \text{Value of } 9.3^{rd} \text{ item} = 130$

$\therefore \quad Q_3 = 130$

$8\left(\frac{N+1}{10}\right)^{th}$

th

$D_8 = \text{Value of item}$

$= \text{Value of } 8\left(\frac{31+1}{10}\right)$item

$= \text{Value of } 24.8^{th} \text{ item} = 150$

$\therefore \quad D_8 = 150$

$70\left(\frac{N+1}{100}\right)^{th}$

th

$P_{70} = \text{Value of}$item

$= \text{Value of } 70\left(\frac{30+1}{100}\right)$item

$= \text{Value of } 21.7^{th} \text{ item} = 150$

$\therefore \quad P_{70} = 150$

(C) CONTINUOUS SERIES

Example 31. ***Calculate Quartiles***, D_6 ***and*** P_{20}

$C.I.$	$0-10$	$10-20$	$20-30$	$30-40$	$40-50$	$50-60$	$60-70$	$70-80$
f	8	22	40	70	90	40	20	10

Solution :

$C.I.$	f	Cumulative Frequency $C \cdot f$

$0-10$	8	8 $(1-8)$
$10-20$	22	30 $(9-30)$
$20-30$	40	70 $(31-70)$
$30-40$	70	140 $(71-140)$
$40-50$	90	230 $(141-230)$
$50-60$	40	270 $(231-270)$
$60-70$	20	290 $(271-290)$
$70-80$	10	300 $(291-300)$
	$\sum f = N = 300$	

Now $Q_1 =$ Size of $\left(\frac{N}{4}\right)^{th}$ item

$= $ Size of $\left(\frac{300}{4}\right)^{th}$ item

$= $ Size of 75^{th} item

$\Rightarrow$ Q_1 lies in the class – interval $30-40$

$\Rightarrow$ $Q_1 = L + \frac{\frac{N}{4} - C\cdot f}{f} \times i$

$= 30 + \frac{75-70}{70} \times 10$

$= 30 + \frac{50}{70} = 30.71$

$Q_1 = 30.71$

$\therefore$

$2\left(\frac{N}{4}\right)^{th}$

$Q_2 =$ Size of item

$= $ Size of $2\left(\frac{300}{4}\right)^{th}$ item

$= $ Size of 150^{th} item

$\Rightarrow$ Q_2 lies in the class – interval $40-50$

$\Rightarrow$ $Q_2 = L + \frac{2\left(\frac{N}{4}\right) - C\cdot f}{f} \times i$

$= 40 + \frac{150-140}{90} \times 10$

$= 40 + \frac{100}{90} = 41.11$

$\therefore \quad Q_2 = 41.11$

$3\left(\frac{N}{4}\right)^{th}$

$Q_3 =$ Size of item

$=$ Size of $3\left(\frac{300}{4}\right)^{th}$ item

$=$ Size of 225^{th} item

$\Rightarrow \quad Q_3$ lies in the class – interval $40 - 50$

$\Rightarrow \quad Q_3 = L + \frac{3\left(\frac{N}{4}\right) - C \cdot f}{f} \times$ i

$= 40 + \frac{225-140}{90} \times 10$

$= 40 + \frac{850}{90} = 49 + 9.44 = 49.44$

$Q_3 = 49.44$

$\therefore$

$6\left(\frac{N}{10}\right)^{th}$

$D_6 =$ Size of item

$=$ Size of $6\left(\frac{300}{10}\right)^{th}$ item

$=$ Size of 180^{th} item

$\Rightarrow \quad D_6$ lies in the class – interval $40 - 50$

$\Rightarrow \quad D_6 = L + \frac{6\left(\frac{N}{10}\right) - C \cdot f}{f} \times$ i

$= 40 + \frac{180-140}{90} \times 10$

$= 40 + \frac{40}{9} = 40 + 4.44 = 44.44$

$D_6 = 44.44$

$\therefore$

$20\left(\frac{N}{100}\right)^{th}$

$P_{20} =$ Size ofitem

$=$ Size of $20\left(\frac{300}{100}\right)^{th}$ item

$=$ Size of 60^{th} item

$\Rightarrow \quad P_{20}$ lies in the class – interval $20 - 30$

$$\Rightarrow \quad P_{20} = L + \frac{20\left(\frac{N}{100}\right) - C.f}{f} \times i$$

$$= 20 + \frac{60-30}{40} \times 10$$

$$= 20 + \frac{30}{4} = 20 + 7.5 = 27.5$$

$$\therefore \quad P_{20} = 27.5$$

7.13 TEST YOUR UNDERSTANDING (C)

1. Calculate Median 30, 45,75, 65, 50, 52, 28, 40, 49, 35, 52,

2. Calculate Median 36, 32,28, 22, 26, 20, 18, 40,

3. Find Median

Wages:	100	150	80	200	250	180	
No. of workers		24	26	16	20	6	30

4. Calculate Median

X;	0-5	5-10	10-15	15-20	20-25	25-30	30-35
F:	4	6	10	16	12	8	4

5. Calculate Median:

X;	10-19	20-29	30-39	40-49	50-59	60-69
F:	4	8	12	16	10	6

6. Find Median:

Income	100-200	200-400	400-700	700-1200	1200-2000
Number of firms	40	100	260	80	20

7. Find missing frequency when median is 50 and number is 100.

X;	0-20	20-40	40-60	60-80	80-100
F:	1 4	?	27	?	15

8. Find Q_1, Q_3, D_5, P_{25} and P_{67}

X: 37, 39, 45, 53, 41, 57, 43, 47, 51, 49, 55

9. Calculate Median, Quartile and D_6

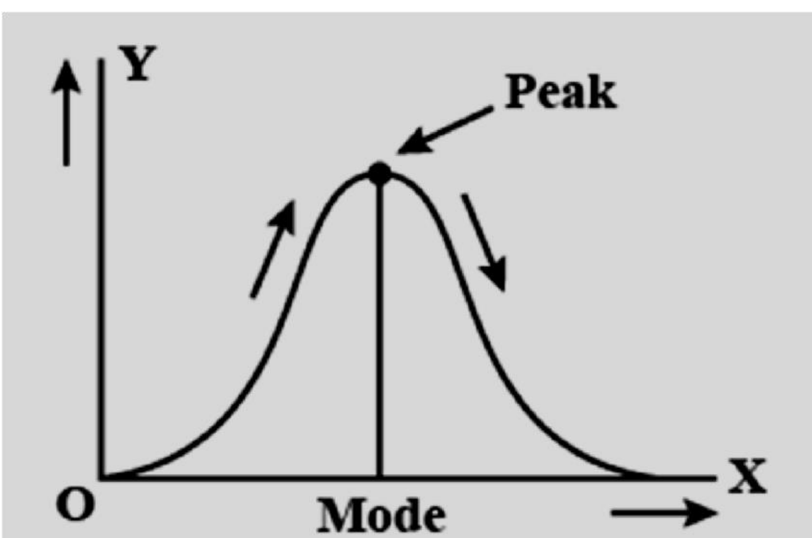
Marks Less Than	80	70	60	50	40	30	20	10
No. of Students	100	90	80	60	32	20	13	5

Answers

1) **49**
2) **27**
3) **150**
4) **18.125**
5) **42**
6) **526.92**
7) **23,21**
8) **41, 53, 47, 41, 51.08**
9) **M = 46.4, Q_1 = 34.2, Q_3 = 57.5, D_6 = 50**

7.14 MODE

Mode is another positional measure of Central Tendency. Mode is the value that is repeated most number of time in the series. In other words the value having highest frequency is called Mode. The term 'Mode' is taken from French word 'La Mode' which means the most fashionable item. So, Mode is the most popular item of the series.



For calculating Mode
1. Series should be in ascending or descending order. **2. Series should be exclusive, not inclusive.** **3. Series should have equal class intervals.**

7.14.1 MODE IN INDIVIDUAL SERIES.

In case of Individual series, following are the steps of finding the Mode.

1. Arrange the series either in ascending order or descending order.
2. Find the most repeated item.
3. This item is Mode.

Example 32. ***Calculate mode from the following data of marks obtained by*** 10 ***students***

S. No.	1	2	3	4	5	6	7	8	9	10
Marks obtained	10	27	24	12	27	27	20	18	15	30

Solution : *By Inspection*

It can be observed that 27 occur most frequently i. e. 3 times. Hence, mode $=$ 27 marks

By converting into discrete series

Marks Obtained	Frequency
10	1
12	1
15	1
18	1
20	1
24	1
27	3
30	1
	$N = 10$

Since, the frequency of 27 is maximum i. e. 3

It implies the item 27 occurs the maximum number of times. Hence the modal marks are 27.

Mode = 27

7.14.2 MODE IN DISCRETE SERIES In case of discrete series, we can find mode by two methods that are Observation Method and Grouping Method.

1. **OBSERVATION METHOD**: Under this method value with highest frequency is taken as mode.
2. Grouping Method: Following are the steps of Grouping method:
    - Prepare a table and put all the values in the table in ascending order.
    - Put all the frequencies in first column. Mark the highest frequency.
    - In second column put the total of frequencies taking two frequencies at a time like first two, than next two and so on. Mark the highest total.
    - In third column put the total of frequencies taking two frequencies at a time but leaving the first frequency like second and third, third and fourth and so on. Mark the highest total.
    - In fourth column put the total of frequencies taking three frequencies at a time like first three, than next three and so on. Mark the highest total.
    - In fifth column put the total of frequencies taking three frequencies at a time but leaving the first frequency like second , third and fourth; than fifth, sixth and seventh and so on. Mark the highest total.
    - In sixth column put the total of frequencies again taking three frequencies at a time but leaving the first two frequencies. Mark the highest total. • Value that is marked highest number of time is the mode.

Example 33. ***Find the modal value for the following distribution***

Age (in years)	8	9	10	11	12	13	14	15
No. of Persons	5	6	8	7	9	8	9	6

Solution: Here, as maximum frequency 9 belongs to two age values 12 and 14, so its not possible to determine mode by inspection. We will have to determine the modal value through grouping and analysis table.

Grouping Table						
Age (in years)	Frequency					
	G_1	G_2	G_3	G_4	G_5	G_6
8	5	11	14	19	21	
9	6					
10	8	15				24

11	7		16	24		
12	9	17			26	23
13	8		17			
14	9	15				
15	6					

Analysis Table

Group No.	8	9	10	11	12	13	14	15
G_1					×		×	
G_2					×	×		
G_3						×	×	
G_4				×	×	×		
G_5					×	×	×	
G_6			×	×	×			
Total	×	×	1	2	5	4	3	×

Since, 12 occurs maximum number of times i. e. 5 times, the modal age is 12 years

Mode= 12

7.14.3 MODE IN CONTINUOUS SERIES

In case of continuous series, we can find mode by two methods that are Observation Method and Grouping Method.

1. **OBSERVATION METHOd**: Under this method value with highest frequency is taken as mode class than the mode formula is applied which is given below.
2. **GROUPING METHOD**: Following are the steps of Grouping method:
    - Prepare a table and put all the classes of data in the table in ascending order.
    - Put all the frequencies in first column. Mark the highest frequency.
    - In second column put the total of frequencies taking two frequencies at a time like first two, than next two and so on. Mark the highest total.
    - In third column put the total of frequencies taking two frequencies at a time but leaving the first frequency like second and third, third and fourth and so on. Mark the highest total.

- In fourth column put the total of frequencies taking three frequencies at a time like first three, than next three and so on. Mark the highest total.
- In fifth column put the total of frequencies taking three frequencies at a time but leaving the first frequency like second , third and fourth; than fifth, sixth and seventh and so on. Mark the highest total.
- In sixth column put the total of frequencies again taking three frequencies at a time but leaving the first two frequencies. Mark the highest total.
- Class that is marked highest number of time is the mode class.
- Apply following formula for calculating the mode:

$$Z = L + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times i$$

Where
Z = Mode
L = Lower limit of the mode class f_m =
Frequency of mode class. f_1 = Frequency of
class preceeding mode class f_2 = Frequency of
class succeeding mode class i = Class interval

Example 34. ***Find the mode for the following frequency distribution***

Age (in years)	$30 - 35$	$35 - 40$	$40 - 45$	$45 - 50$	$50 - 55$	$55 - 60$
No. of Persons	3	8	12	20	15	2

Solution : Here, the maximum frequency is corresponding to the class – interval $45 - 50$.

So, the modal class is $45 - 50$.

Now, the mode is given by the formula

Mode, $Z = L + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times i$

Here L = Lower limit of modal class = 45

f_m = Frequency of modal class = 20

f_1 = Frequency of class preceeding the modal class = 12

f_2 = Frequency of class succeeding the modal class = 15 i

= Class length of modal class = 5

∴ Mode, $Z = 45 + \frac{20-12}{(2\times 20)-12-15} \times 5$

$= 45 + \frac{8}{40-27} \times 5$

$= 45 + 3.07$

= 48.1 years (approx.)

$\Rightarrow \quad Z = 48.1$ years

Example 35. ***Calculate mode from the following data***

$C.I.$	***0-10***	***10-20***	***20-30***	***30-40***	***40-50***	***50-60***	***60-70***	***70-80***	***80-90***	***90-100***
f	2	9	10	13	11	6	13	7	4	1

Solution : Here as it is not possible to find modal class by inspection, so we have to determine it through grouping and analysis table.

<table>
<tr><th colspan="7">Grouping Table</th></tr>
<tr><th rowspan="2">$C.I.$</th><th colspan="6">Frequency</th></tr>
<tr><th>G_1</th><th>G_2</th><th>G_3</th><th>G_4</th><th>G_5</th><th>G_6</th></tr>
<tr><td>$0-10$</td><td>2</td><td rowspan="2">11</td><td></td><td rowspan="3">21</td><td></td><td></td></tr>
<tr><td>$10-20$</td><td>9</td><td rowspan="2">19</td><td rowspan="3">32</td><td></td></tr>
<tr><td>$20-30$</td><td>10</td><td rowspan="2">23</td><td rowspan="3">34</td></tr>
<tr><td>$30-40$</td><td>13</td><td rowspan="2">24</td><td rowspan="3">30</td></tr>
<tr><td>$40-50$</td><td>11</td><td rowspan="2">17</td><td rowspan="3">30</td></tr>
<tr><td>$50-60$</td><td>6</td><td rowspan="2">19</td><td rowspan="3">26</td></tr>
<tr><td>$60-70$</td><td>13</td><td rowspan="2">20</td><td rowspan="3">24</td></tr>
<tr><td>$70-80$</td><td>7</td><td rowspan="2">11</td><td rowspan="3">12</td></tr>
<tr><td>$80-90$</td><td>4</td><td rowspan="2">5</td><td></td></tr>
<tr><td>$90-100$</td><td>1</td><td></td><td></td><td></td></tr>
</table>

Analysis Table

Group No.	***0-10***	***10-20***	***20-30***	***30-40***	***40-50***	***50-60***	***60-70***	***70-80***	***80-90***	***90-100***
G_1				×			×			
G_2			×	×						
G_3				×	×					
G_4				×	×	×				
G_5		×	×	×						
G_6			×	×	×					
Total	×	1	3	6	3	1	1	×	×	×

Clearly the modal class is $30-40$

Now the mode is given by the formula

Mode, $Z = L + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times \text{i}$

Here L = Lower limit of modal class $30 - 40 = 30$ f_m = Frequency corresponding to modal class = 13 f_1 = Frequency of interval preceding modal class f_2 = Frequency of interval succeeding and i = Class length of modal class

∴ Mode, $Z = 30 + \frac{13-10}{(2\times13)-10-11} \times 10$

$= 30 + \frac{3}{26-21} \times 10$

$= 30 + \frac{30}{5}$

$= 30 + 6$

$= 36$

⇒ $Z = 36$

Example 36. ***Determine the missing frequencies when it is given that*** $N = 230$***, Median*** M***,*** $= 233.5$ ***and Mode,*** $Z = 234$

C.I	***200-210***	***210-220***	***220-230***	***230-240***	***240-250***	***250-260***	***260-270***

f	4	16	−	−	−	6	4

Solution : Let the missing frequencies be f_1, f_2 and f_3 respectively.

$C.I$	f	$C \cdot f$
$200 - 210$	4	4
$210 - 220$	16	20
$220 - 230$	f_1	$20 + f_1$
$230 - 240$	f_2	$20 + f_1 + f_2$
$240 - 250$	f_3	$20 + f_1 + f_2 + f_3$
$250 - 260$	6	$26 + f_1 + f_2 + f_3$
$260 - 270$	4	$30 + f_1 + f_2 + f_3$
	$N = 230 = \sum f$ $\sum f = 30 + f_1 + f_2 + f_3$	

Now $N = 230 = \sum f$ (Given)

$= 30 + f_1 + f_2 + f_3$

$\Rightarrow$ $f_1 + f_2 + f_3 = 230 - 30 = 200$

$\Rightarrow$ $f_1 + f_2 + f_3 = 200$ …(i)

Also, Median $= 233.5$ (Given)

$\Rightarrow$ Median class is $230 - 240$

$$\Rightarrow \quad M = L + \frac{\frac{N}{2} - C \cdot f}{f} \times i$$

$$233.5 = 230 + \frac{\frac{230}{2} - (20 + f_1)}{f_2} \times 10$$

$$3.5 = \frac{115 - 20 - f_1}{f_2} \times 10$$

$$3.5f_2 = 950 - 10f_1$$

$$10f_1 + 3.5f_2 = 950$$

$\Rightarrow$ …(ii)

Now Mode $= 234$ lies in $230 - 240$

$$\therefore \quad Z = L + \frac{f_2 - f_1}{2f_2 - f_1 - f_3} \times i$$

$\Rightarrow \quad 234 = 230 + \frac{f_2 - f_1}{2f_2 - f_1 - f_3} \times 10$

$\Rightarrow \quad 4 = \frac{f_2 - f_1}{2f_2 - f_1 - (200 - f_1 - f_2)} \times 10$ [Using (i)]

$4 = \frac{f_2 - f_1}{2f_2 - f_1 - 200 - f_1 - f_2} \times 10$

$\Rightarrow$

$\Rightarrow \quad 4 = \frac{(f_2 - f_1) \times 10}{3f_2 - 200}$

$\Rightarrow \quad 12f_2 - 800 = 10f_2 - 10f_1$

$\Rightarrow \quad 2f_2 - 800 + 10f_1 = 0$

$\Rightarrow \quad 10f_1 + 2f_2 = 800$ …(iii)

Solving (ii) and (iii), we get

$$\begin{aligned} 10f_1 + 3.5f_2 &= 950 \\ 10f_1 + 2f_2 &= 800 \\ (-)(-)\quad (-) & \\ \hline 1.5f_2 &= 150 \end{aligned}$$

$\Rightarrow \quad f_2 = \frac{150}{1.5} = 100$

$f_2 = 100$

…(iv)

Put (iv) in (iii)

$10f_1 + 2(100) = 800$

$\Rightarrow \quad 10f_1 = 800 - 200 = 600$

$\Rightarrow \quad 10f_1 = 600$

$\Rightarrow \quad f_1 = 60$ …(v)

Put (iv) and (v) in (i)

$60 + 100 + f_3 = 200$

$\Rightarrow \quad f_3 = 40$

∴ The missing frequencies are 60, 100 and 40.

7.14.4 MERITS OF MEDIAN

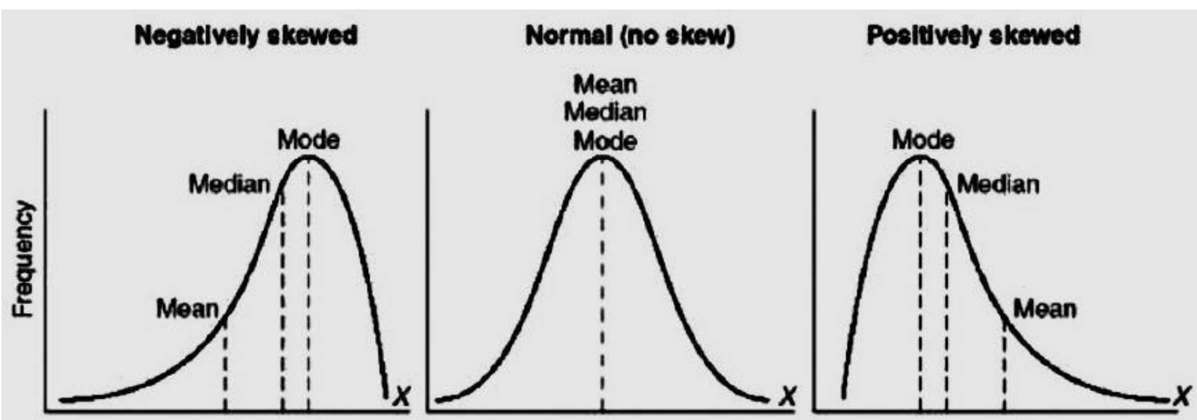
1. Mode is easy to calculate.
2. People can understood this in routine life.
3. It is capable of Graphic presentation.
4. It is possible even in case of open end series.
5. This is rigidly defined.
6. It is not affected by extreme values.
7. In case of qualitative data, it is very useful**.**

7.14.5 LIMITATIONS OF MEDIAN

1. It is not always determinable as series may be Bi-modal or Tri-modal.
2. It is not capable of further algebraic treatment.
3. It is positional average and is not based on all observation.
4. It is very much affected by fluctuation in sampling. 5. Mode needs arrangement of data before calculation.

7.15 RELATION BETWEEN MEAN, MEDIAN AND MODE

In a normal series the value of Mean, Median and Mode is always same. However, Karl Pearson studied the empirical relation between the Mean, Median and Mode and found that in moderately skewed series the Median always lies between the Mean and the Mode. Normally it is two third distance from Mode and one third distance from Mean.



On the basis of this relation following formula emerged

Mode = 3 Median – 2 Mean or

$Z = 3M - 2\overline{X}$

Example. ***Calculate*** M ***when*** $\overline{X}$ ***and*** Z ***of a distribution are given to be*** 35. 4 ***and*** 32. 1 ***respectively.***

Solution : We are given that

Mean, $\overline{X} = 35.4$
Mode, $Z = 32.1$

As we know the empirical relation between Mean, Median and Mode.

i. e. Mode = 3 Median – 2 Mode

$\Rightarrow$ $Z = 3M - 2\overline{X}$

$M = \frac{1}{3}(Z + 2\overline{X})$

$M = \frac{1}{3}(32.1 + 2(35.4))$

$= \frac{1}{3}(32.1 + 70.8)$

$= \frac{1}{3}(102.9) = 34.3$

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$ Median, $M = 34.3$

7.16 TEST YOUR UNDERSTANDING - D

1. Find Mode:
X 22, 24, 17, 18, 19, 18, 21, 20, 21, 20, 23, 22, 22, 22

2. Find Mode by inspection method

X	6	12	18	24	30	36	42	48
f 9	11	25	16	9		10	6	3

3. . Find Mode by Grouping Method

X	21	22	25	26	27	28	29	30

F	7	10	15	18	13	7	3	2

4. . Find Mode by Grouping Method and inspection method

X;	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
F:	2	18	30	45	35	20	6	4

5. Calculate mode using grouping and analysis methods.

X	100-110	110-120	120-130	130-140	140-150	150-160	160-170	170-180
f	4	6	20	32	33	17	8	2

6. Find Mode

X	0-100	100-200	200-400	400-500	500-700
F:	5	15	40	32	28

Anwers.

1. 22
2. 18
3. 26
4. 36
5. 56.46
6. 440

7.17 LET US SUM UP

- **Average is the value that represent its series.**
- **A good average has many characteristics.**
- **Average is also known as Central Tendency.**
- **There are mainly five types of average Arithmetic Mean, Geometric Mean, Harmonic Mean, Median, Mode.**
- **Arithmetic mean is most popular average.**
- **Geometric Mean is useful for calculating growth and decline rate.**
- **Harmonic mean is useful for speed and work etc.**
- **Median divide the series in two equal parts.**
- **Mode is value repeated most number of time.**
- **There are other positional measures like Quartile, Decile and Percentile.**

7.18 KEY TERMS

- **Average:**Average is the single value which is capable of representing its series. It is the value around which other values in the series move.
- **Arithmetic Mean:** " The value obtained by dividing sum of observations with the number of observations".
- **Geometric Mean:** Geometric mean of N items is root nth of the product of item • **Harmonic Mean** . It is reciprocal of Arithmetic mean of reciprocal of the observations.
- **Median:** It is a value that divide the series in two equal parts.
- **Mode:** It is the most repeated value of the series.
- **Quartile:** It is a value that divide the series in four equal parts.
- **Decile:** It is a value that divide the series in ten equal parts.
- **Percentile:** It is a value that divide the series in hundred equal parts.

7.19 REVIEW QUESTIONS

1. What is central tendency. What are uses of measuring central tendency.
2. Give features of ideal measure of average.
3. What is average. Give uses and limitations of average.
4. What is arithmetic mean? How it is calculated.
5. Give properties, advantages and limitations of Arithmetic mean.
6. How you can calculated combined arithmetic mean.
7. What is median? How it is calculated?
8. Give merits and limitations of Median.
9. What is mode? How it is calculated. Give its merits and limitations.
10. Explain grouping method of calculating Mode.
11. What is Geometric Mean. Give process of calculating Geometric mean in different series.
12. What are merits and limitations of geometric mean.
13. What is Harmonic Mean. How it is calculated.
14. Give relation between Mean, Median and Mode.
15. What are Quartile, Percentile and Deciles.
16. What is positional average. Give various positional average. 17. According to you which measure of average is best.

7.20 FURTHER READINGS

1. J. K. Sharma, *Business Statistics*, Pearson Education.
2. S.C. Gupta, *Fundamentals of Statistics*, Himalaya Publishing House.
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B. COM (DIGITAL)

SEMESTER II

COURSE: BUSINESS MATHEMATICS AND STATISTICS

UNIT 8 – DISPERSION

STRUCTURE

8.0 OBJECTIVES

After studying the Unit, students will be able to

- Explain what is Dispersion.
- Compare absolute and relative measures of Dispersion.
- Understand features of good measure of Dispersion.
- Calculate the Range and Quartile Deviation.
- Measure the Dispersion using Mean and Standard Deviation. • Compare the variation of the two series.

8.1 INTRODUCTION

Statistics is a tool that helps us in extraction of information from large pool of data. There are many tools in statistics that helps us in extraction of data. Central tendency of data is one such tool. A good measure of central tendency is one which could represent the whole data. However, many a time we find that the average is not representing it data. Following example will make this clear:

Series X	Series Y	Series Z
100	94	1

100 100 100 100	105 101 98 102	2 3 4 490
$\sum X = 500$	$\sum Y = 500$	$\sum Z = 500$
$\bar{X} = \frac{\sum X}{N} = \frac{500}{5} = 100$	$\bar{Y} = \frac{\sum Y}{N} = \frac{500}{5} = 100$	$\bar{Z} = \frac{\sum Z}{N} = \frac{500}{5} = 100$

We can see that in all the above series the average is 100. However, in first series average is fully representing its data as all the items in the series are 100 and average is also 100. In the second series the items are very near to its average that is 100, so we can say that average is is a good representation of its series. But in case of third series, average is not representing its data as there is a lot of difference between items and the average. In order to understand the nature of data it is very important to see the difference between items and the data. This could be done by using dispersion.

8.2 MEANING OF DISPERSION

Dispersion is a very important statistical tool that help us in understanding the nature of data. Dispersion shows the extent to which individual items in the data differs from its average. It is a measure of difference between data and the individual items. It indicates that how that are lacks the uniformity. Following are some of the definitions of Dispersion.

According to Simpson and Kafka, "The measures of the scatterness of a mass of figures in a series about an average is called measure of variation, or dispersion".

According to Spiegel, "The degree to which numerical data lend to spread about an average value is called the variation, or dispersion of the data".

As the dispersion gives average of difference between items and its Central tendency, it is also known as average of second order.

8.3 BENEFITS / USES OF DISPERSION

Benefits of Dispersion analysis are outlined as under :

1. **To examine reliability of Central tendency:** We have already discussed that a good measure of Central tendency is one which could represent its series. Dispersion gives us the idea that whether average is in a position to represent its series or not. On the basis of this we can calculate reliability of the average.
2. **To compare two series**: In case there are two series and we want to know that which series is having more variation, we can use dispersion as it tool. In such case normally we use relative measure of dispersion for comparing two series.

3. **Helpful in quality control**: Dispersion is tool which is frequently used in quality control by the business houses. Every manufacturer wants to maintain same quality and reduce the variation in production. Dispersion can help us in finding the deviations and removing the deviations in quality.
4. **Base of further statistical analysis:** Dispersion is a tool that is used in a number of statistical analysis. For example we use dispersion while calculating correlation, Regression, Skewness and Testing the Hypothesis etc.

8.4 FEATURES OF GOOD MEASURE OF DISPERSION

A good measure of dispersion has a number of features which are mentioned below:

1. A good tool of dispersion must be easy to understand and simple to calculate.
2. A good measure of dispersion must be based on all the values in the data.
3. It should not been affected by presence of extreme values in the data.
4. A good measure is one which is rigidly defined.
5. A good measure of dispersion must be capable of being further statistical analysis. 6. A good measure must not be affected by the sampling size.

8.5 ABSOLUTE AND RELATIVE MEASURE OF DISPERSION

There are two measures of dispersion that are absolute measure and relative measure:

1. **Absolute measure:** the absolute measure of dispersion is one which is expressed in the same statistical unit in which the original values of that data are expressed. For example if original data is represented in kilograms, the dispersion will also be represented in kilogram. Similarly if data is represented in rupees the dispersion will also be represented in rupees. However, this measure is not useful when we have to compare two or more series that are having different units of measurement or belongs to different population.
2. **Relative measure of Dispersion**: The relative measure of dispersion is independent of unit of measurement and is expressed in pure number. Normally it is a ratio of the dispersion to the average of the data. It is very useful when we have to compare two different series that are having different unit of measurement or belongs to different population.

Absolute Measure of Dispersion	Relative Measure of Dispersion
1. Range	1. Coefficient of Range
2. Quartile Deviation	2. Coefficient of Quartile Deviation
3. Mean Deviation	3. Coefficient of Mean Deviation
4. Standard Deviation	4. Coefficient of Standard Deviation

8.6 MEASURE OF DISPERSION - RANGE

Range is one of the simplest and oldest measure of Dispersion. We can define Range as the difference between highest value of the data and the lowest value of the data. The more is the difference between highest and the lowest value, more is the value of Range which shows high dispersion. Similarly less is the difference between highest and lowest value, less is value of Range which shows less dispersion. Following is formula for calculating the value of range:

Range = Highest Value - Lowest Value
R = H – L

Coefficient of Range: Coefficient of Range is relative measure of Range and can be calculated using the following formula.

$$\textbf{Coefficient of Range} = \frac{Highest\ Value - Lowest\ Value}{Highest\ Value + Lowest\ Value} = \frac{H - L}{H + L}$$

8.6.1 Range in Individual Series:

Example 1. Following are daily wages of workers, find out value of Range and Coefficient of Range.

Wage (Rs.)	330	300	470	500	410	380	425	360

Solution:

Range = Highest Value – Lowest Value

$= 500 – 300$
$= 200$

$$\text{Coefficient of Range} = \frac{Highest\ Value - Lowest\ Value}{Highest\ Value + Lowest\ Value}$$
$$= \frac{500 - 300}{500 + 300}$$
$$= .25$$

8.6.2 Range in Discrete Series:

Example 2. Following are daily wages of workers, find out value of Range and Coefficient of Range.

Wage (Rs.)	300	330	360	380	410	425	470	500
No. of Workers	5	8	12	20	18	15	13	9

Solution:

Range = Highest Value − Lowest Value

= 500 – 300

= 200

Coefficient of Range $= \frac{\text{ighest Value} - \text{Lowest Value}}{\text{Highest Value} + \text{Lowest Value}}$

$= \frac{500 - 300}{500 + 300}$

= .25

8.6.3 Range in Continuous Series:

Example 3. Following are daily wages of workers, find out value of Range and Coefficient of Range.

Wage (Rs.)	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90
No. of Workers	5	8	12	20	18	15	13	9

Solution:

Range = Highest Value − Lowest Value Coefficient

= 90 – 10

= 80

of Range $= \frac{\text{Highest Value} - \text{Lowest Value}}{\text{Highest Value} + \text{Lowest Value}}$

$= \frac{90 - 10}{90 + 10}$

= .80

8.6.4 Merits of Range

1. Range is one of the easiest and simplest method of dispersion.
2. The range a measure that is rigidly defined.
3. This method gives broad picture of variation in the data.
4. Range is very useful in various fields of business such as quality control and checking the difference between share prices in the stock exchange. 5. Range is also useful in forecasting.

8.6.5 Limitations of range

1. Rangee is not exact measure of depreciation at only gives vague picture.
2. It is not based on all the values of data.
3. It is affected by the extreme values of the data.
4. It is also affected by fluctuations in the sample. 5. In case of open-ended series range cannot be calculated.

8.7 TEST YOUR UNDERSTANDING (A)

.1. Compute for the following data Range and Coefficient of Range

28	110	27	77	19	94	63	25	111

2. Given below is heights of students of two classes. Compare Range of the heights :

Class I 167 162 155 180 182 175 185 158 Class II
169 172 168 165 177 180 195 167

3. Find Range and coefficient of Range

X	5	10	15	20	25	30	35	40
f	6	4	12	7	24	21	53	47

4. Calculate coefficient of Range:

X;	10-20	20-30	30-40	40-50	50-60
F:	8	10	12	8	4

Answers

1. **92, 0.7**
2. **.088, .083**
3. **35, 0.778**
4. **.714**

8.8 MEASURE OF DISPERSION – QUARTILE DEVIATION

Range is simple to calculate but suffers from limitation that it takes into account only extreme values of the data and gives a vague picture of variation. Moreover it cannot be calculated in case of open end series. In such case we can use another method of Deviation that is Quartile Deviation or Quartile Range. Quartile Range is the difference between Third Quartile and First Quartile of the data. Following is formula for calculating Quartile Range.

Quartile Range = $Q_3 – Q_1$

Quartile Deviation: Quartile deviation is the Arithmetic mean of the difference between Third Quartile and the First Quartile of the data.

Quartile Deviation $= \frac{Q_3 - Q_1}{2}$

Coefficient of Quartile Deviation: Coefficient of Quartile Deviation is relative measure of Quartile Deviation and can be calculated using the following formula.

Coefficient of Range $= \frac{Q_3 - Q_1}{Q_3 + Q_1}$

8.8.1 QUARTILE DEVIATION IN INDIVIDUAL SERIES:

Example 4. Following are daily wages of workers, find out value of Quartile Range, Quartile Deviation and Coefficient of Quartile Deviation.

Wage (Rs.)	300	330	380	410	425	470	500

Solution:

Q_1 = Value of $\frac{N+1}{4}$th item = Value of $\frac{7+1}{4}$th item

= Value of 2n item

= 330

Q3 = Value of $\frac{3(N+1)}{4}$th item = Value of $\frac{3(7+1)}{4}$th item

= Value of 6th item

= 470

Quartile Range = $Q_3 – Q_1$

= 470 – 330

= 140

Quartile Deviation $= \frac{Q3 - Q1}{2}$

$= \frac{470 - 330}{2}$

= 70

Coefficient of Quartile Deviation $= \frac{Q3 - Q1}{Q_3 + Q_1}$

$= \frac{470 - 330}{470 + 330}$

= .175

8.8.2 QUARTILE DEVIATION IN DISCRETE SERIES:

Example 5. Following are daily wages of workers, find out value of Quartile Range, Quartile Deviation and Coefficient of Quartile Deviation.

Wage (Rs.)	300	330	380	410	425	470	500
No. of Workers	5	8	12	20	18	15	13

Solution:

Calculation of Quartile

Wage (Rs.) (X)	**No. of Workers (f)**	**Cumulative Frequency (cf)**
300	5	5
330	8	13
380	12	25
410	20	45
425	18	63
470	15	78
500	13	91

$Q_1 =$ Value of $\frac{N+1}{4}$th item = Value of $\frac{91+1}{4}$th item

= Value of 23rd item

= 380

$Q3 =$ Value of $\frac{3(N+1)}{4}$th item = Value of $\frac{3(91+1)}{4}$th item

= Value of 69th item

= 470

Quartile Range = $Q_3 - Q_1$

= 470 – 380

= 90

Quartile Deviation $= \frac{Q3 - Q1}{2}$

$= \frac{470 - 380}{2}$

= 45

Coefficient of Quartile Deviation $= \frac{Q3 - Q1}{Q_3 + Q_1}$

$= \frac{470 - 380}{470 + 380}$

= .106

8.8.3 QUARTILE DEVIATION IN CONTINUOUS SERIES:

Example 6. Following are daily wages of workers, find out value of Quartile Range, Quartile Deviation and Coefficient of Quartile Deviation.

Wage (Rs.)	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90
No. of Workers	5	8	12	20	18	15	13	9

Solution:

Calculation of Quartile

Wage (Rs.) (X)	**No. of Workers (f)**	**Cumulative Frequency (cf)**
10-20	5	5
20-30	8	13
30-40	12	25
40-50	20	45
50-60	18	63
60-70	15	78
70-80	13	91
80-90	9	100

Calculation of Q_1

Q_1 Class = Value of $\frac{N}{4}$th item = Value of $\frac{100}{4}$th item

Q_1 Class = Value of 25th item

Q_1 Class = 30-40

$$Q_1 = L_1 + \frac{\frac{n}{4} - cf}{f} \times c$$

Where $L_1 = 30$, $n = 100$; $cf = 13$; $f = 12$; $c = 10$

$$Q_1 = 30 + \frac{\frac{100}{4} - 13}{12} \times 10 = 40$$

Calculation of Q_3

Q_3 Class = Value of $\frac{3N}{4}$ th item = Value of $\frac{300}{4}$th item

Q_3 Class = Value of 75th item

Q_3 Class = 60-70

$$Q_3 = L_1 + \frac{\frac{3n}{4} - cf}{f} \times c$$

Where $L_1 = 60$, $n = 100$; $cf = 63$; $f = 15$; $c = 10$

$$Q_1 = 60 + \frac{\frac{3(100)}{4} - 63}{15} \times 10 = 68$$

Calculation of Quartile Range, Quartile Deviation and Coefficient of Quartile Deviation

Quartile Range = $Q_3 - Q_1$

= 68 – 40

= 28

Quartile Deviation $= \frac{Q3 - Q1}{2}$

$= \frac{68 - 40}{2}$

= 14

Coefficient of Quartile Deviation $= \frac{Q3 - Q1}{Q3 + Q1}$

$$= \frac{68 - 40}{68 + 40}$$

$= .259$

8.8.4 MERITS OF QUARTILE DEVIATION

1. Quartile deviation is a tool which is easy to calculate and understand.
2. Quartile deviation is the best tool of dispersion in case of open ended series.
3. This method of dispersion is better than range.
4. Unlike the range, it is not affected by the extreme values.
5. This method of dispersion is rigidly defined.
6. This method is very useful specially when we want to know the variability of middle half of the data. Under this method first 25% items that are less than Q_1 and upper 25% items that are more than Q_3 are excluded and only middle 50% items are taken.

8.8.5 LIMITATIONS OF QUARTILE DEVIATION

1. Quartile deviation considers only middle 50% items of the data and ignore rest of the items.
2. It is not possible to make any further algebraic treatment of the quartile deviation.
3. It is not based on all the items.
4. Quartile deviation is highly affected by fluctuation in the sample.
5. It is comparatively difficult to calculate quartile deviation than range.

8.9 TEST YOUR UNDERSTANDING (B)

1. Find Quartile deviation and coefficient of Quartile Deviation:
   X: 59, 60, 65, 64, 63, 61, 62, 56, 58, 66

2. Find Quartile deviation and coefficient of Quartile Deviation::

X	58	59	60	61	62	63	64	65	66
F	15	20	32	35	33	22	20	10	8

3. Find Quartile deviation and coefficient of Quartile Deviation

X	0-100	100-200	200-300	300-400	400-500	500-600	600-700
F:	8	16	22	30	24	12	6

4. Calculate Inter Qualtile Range, Q.D and coefficient of Q.D

X	0-10	10-20	20-30	30-40	0-500	50-60	60-70	70-80	80-90
F:	11	18	25	28	30	33	22	15	22

Answers

1. **2.75, 0.0447**
2. **1.5, .024**
3. **113.54, 0.335**
4. **34.84, 17.42, .3769**

8.10 MEASURE OF DISPERSION – MEAN DEVIATION

Both Range and Quartile Deviation are positional method of Dispersion and takes into consideration only two values. Range considers only highest and lowest value while calculating Dispersion, while Quartile Deviation considers on First and Third Quartile for calculating Dispersion. Both these methods are not based on all the values of the data and are considerable affected by the sample unit. A good measure of Dispersion is one which considers all the values of data.

Mean Deviation is a tool of measuring the Dispersion that is based on all the values of Data. Contrary to its name, it is not necessary to calculate Mean Deviation from Mean, it can also be calculated using the Median of the data or Mode of the data. In the Mean deviation we calculated deviations of the items of data from its Average (Mean, Median or Mode) by taking positive signs only. When we divide the sum of deviation with the number of items, we get the value of Mean Deviation. In simple words:

"Mean Deviation is the value obtained by taking arithmetic mean of the deviations obtained by deducting average of data whether Mean, Median or Mode from values of data, ignoring the signs of the deviations."

8.10.1 MEAN DEVIATION IN CASE OF INDIVIDUAL SERIES:

As we have already discussed that Mean Deviation can be calculated from Mean, Median or Mode. Following are the formula for calculating Mean Deviation in case of Individual series.

Mean Deviation from Mean $(MD_{\bar{X}}) = \frac{\sum |X - \bar{X}|}{n} = \frac{\sum |D_{\bar{X}}|}{n}$

Mean Deviation from Median $(MD_M) = \frac{\sum |X - M|}{n} = \frac{\sum |D_M|}{n}$

Mean Deviation from Mode $(MD_Z) = \frac{\sum |X - Z|}{n} = \frac{\sum |D_Z|}{n}$

In case we want to calculate Coefficient of Mean Deviation, it can be done using following formulas.

Coefficient of Mean Deviation from Mean $(MD_{\bar{X}}) = \frac{MD_{\bar{X}}}{\bar{X}}$

Coefficient of Mean Deviation from Median (MD_M) = $\frac{MD_M}{M}$

Coefficient of Mean Deviation from Mode (MD_Z) = $\frac{MD_Z}{Z}$

Example 7. Following are the marks obtained by Students of a class in a test. Calculated Mean Deviation from (i) Mean (ii) Median (iii) Mode. Also calculate Coefficient of Mean Deviation.

Wage (Rs.)	5	7	8	8	9	11	13	14	15

SOLUTION:

Let us calculate Mean Median and Mode

Mean $(\bar{X}) = \frac{5+7+8+8+9+11+13+14+15}{9} = \frac{90}{9} = 10$

Median (M) = Value of $\frac{N+1}{2}$th item = Value of $\frac{9+1}{2}$th item

= Value of 5th item

= 9 Mode = Item having maximum frequency i.e 8.

Calculation of Deviations

Marks X	$D_{\bar{X}} = \lvert X - \bar{X} \rvert$ **(Where $\bar{X}$ = 10)**	$D_M = \lvert X - M \rvert$ **(Where M = 9)**	$D_Z = \lvert X - Z \rvert$ **(Where Z = 8)**
5	5	4	3
7	3	2	1
8	2	1	0
8	2	1	0
9	1	0	1
11	1	2	3
13	3	4	5
14	4	5	6
15	5	6	7
	$\sum D_{\bar{X}}$ = **26**	$\sum D_M$ = **25**	$\sum D_Z$ = **26**

1. Mean Deviation from Mean $(MD_{\bar{X}}) = \frac{\sum \lvert X-\bar{X} \rvert}{n} = \frac{\sum \lvert D_{\bar{X}} \rvert}{n} = \frac{26}{9} = 2.88$

Coefficient of Mean Deviation from Mean $(MD_X) = \frac{MD_X}{\bar{X}} = \frac{2.88}{10} = .288$

2. Mean Deviation from Median $(MD_M) = \frac{\sum | X - M |}{n} = \frac{\sum | DM |}{n} = \frac{25}{9} = 2.78$

Coefficient of Mean Deviation from Median $(MD_M) = \frac{MDM}{M} = \frac{2.78}{9} = .309$

3. Mean Deviation from **Mode** $(MD_Z) = \frac{\sum | X - Z |}{n} = \frac{\sum | DZ |}{n} = \frac{26}{9} = 2.88$

Coefficient of Mean Deviation from **Mode** $(MD_Z) = \frac{MDZ}{Z} = \frac{2.88}{8} = .36$

8.10.2 MEAN DEVIATION IN CASE OF DISCRETE SERIES:

Following are the formula for calculating Mean Deviation in case of Discrete series.

Mean Deviation from Mean $(MD_X) = \frac{\sum f | X - \bar{X} |}{n} = \frac{\sum f | D_X |}{n}$
Mean Deviation from Median $(MD_M) = \frac{\sum f | X - M |}{n} = \frac{\sum f | DM |}{n}$
Mean Deviation from Mode $(MD_Z) = \frac{\sum f | X - Z |}{n} = \frac{\sum f | D_Z |}{n}$

Example 8. Following are the wages of workers that are employed in a factory. Calculate Mean Deviation from (i) Mean (ii) Median (iii) Mode. Also calculate Coefficient of Mean Deviation.

Wage (Rs.)	300	330	380	410	425	470	500
No. of Workers	6	8	15	25	18	15	13

Solution:

Let us calculate Mean Median and Mode

X	**f**	**fX**	**Cf**
300	5	1500	5
330	8	2640	13
380	15	5700	28
410	26	10660	54
425	18	7650	72
470	15	7050	87
500	13	6500	100
		$\sum X$ = 41700	

Mean $(\bar{X}) = \frac{\sum X}{n} = \frac{41700}{100} = 417$

Median (M) = Value of $\frac{N+1}{2}$th item = Value of $\frac{100+1}{2}$th item

= Value of 50.5 item

= 410

Mode = Item having maximum frequency i.e 410.

Calculation of Deviations

X	f	$D_{\bar{X}} = \lvert X - \bar{X} \rvert$ ($\bar{X}$ = 417)	$fD_{\bar{X}}$	$D_M = \lvert X - M \rvert$ (M = 410)	fD_M	$D_Z = \lvert X - Z \rvert$ (Z = 410)	fD_Z
300 330 380 410 425 470 500	5 8 15 26 18 15 13	117 87 37 7 8 53 83	585 696 555 182 144 795 1079	110 80 30 0 15 60 90	550 640 450 0 270 900 1170	110 80 30 0 15 60 90	550 640 450 0 270 900 1170
			$\sum fD_{\bar{X}}$ =		$\sum fD_M$ =	$\sum D_Z$ = 26	$\sum fD_Z$ =
			4036		3980		3980

1. Mean Deviation from Mean $(MD_{\bar{X}}) = \frac{\sum f \lvert X - \bar{X} \rvert}{n} = \frac{\sum f \lvert D_{\bar{X}} \rvert}{n} = \frac{4036}{100} = 40.36$

Coefficient of Mean Deviation from Mean $(MD_{\bar{X}}) = \frac{MD_{\bar{X}}}{\bar{X}} = \frac{40.36}{417} = .097$

2. Mean Deviation from Median $(MD_M) = \frac{\sum f \lvert X - M \rvert}{n} = \frac{\sum f \lvert D_M \rvert}{n} = \frac{3980}{100} = 39.80$

Coefficient of Mean Deviation from Median $(MD_M) = \frac{MD_M}{M} = \frac{39.80}{410} = .097$

3. Mean Deviation from Mode $(MD_Z) = \frac{\sum f \lvert X - Z \rvert}{n} = \frac{\sum f \lvert D_Z \rvert}{n} = \frac{3980}{100} = 39.80$

Coefficient of Mean Deviation from Mode $(MD_Z) = \frac{MD_Z}{Z} = \frac{39.80}{410} = .097$

8.10.3 MEAN DEVIATION IN CASE OF CONTINUOUS SERIES:

In case of calculation of Mean Deviation in continuous series, the formula will remain same as we have done in Discrete Series but only difference is that instead of taking deviation from Data, we take deviations from mid value of the data. Further in case of continuous series also the Mean Deviation can be calculated from Mean, Median or Mode. However, in most of the cases it is calculated from Median. Following formulas are used for continuous series:

$$(M\ D_{\bar{X}}) = \frac{\sum f|X-\bar{X}|}{n} = \frac{\sum f|D_{\bar{X}}|}{n}$$

Mean Deviation from Mean

$$(M\ D_M) = \frac{\sum f|X-M|}{n} = \frac{\sum f|D_M|}{n}$$

Mean Deviation from Median

Mean Deviation from Mode $(M\ D_Z) = \frac{\sum f|X-Z|}{n} = \frac{\sum f|D_Z|}{n}$

Example 9. Following are daily wages of workers, find out value of Mean Deviation and Coefficient of Mean Deviation.

Wage (Rs.)	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90
No. of Workers	5	8	12	20	18	15	13	9

Solution:

Wage (Rs.) (X)	**No. of Workers (f)**	**Cumulative Frequency (cf)**	**Mid Value m**	$\lvert D_M \rvert$ $\lvert m - M \rvert$	$\lvert f D_M \rvert$
10-20	5	5	15	37.78	188.9
20-30	8	13	25	27.78	222.24
30-40	12	25	35	17.78	213.36
40-50	20	45	45	7.78	155.6
50-60	18	63	55	2.22	39.96
60-70	15	78	65	12.22	183.3
70-80	13	91	75	22.22	288.86
80-90	9	100	85	32.22	289.98
	N = 100				$\sum \lvert f D_M \rvert$ **= 1582.2**

<u>Calculation of Median</u>

Median Class = Value of $\frac{N}{2}$th item = Value of $\frac{100}{2}$th item

Median Class = Value of 50th item Median Class = 50-60

$$= L_1 + \frac{\frac{n}{2} - cf}{f}$$

M × c

Where $L_1 = 50$, n = 100; cf = 45; f = 18; c = 10

$$M = 50 + \frac{\frac{100}{2} - 45}{18} \times 10 = 52.78$$

<u>Calculation of Mean Deviation from Median</u>

Mean Deviation from Median $(MD_M) = \frac{\Sigma f| X - M |}{n} = \frac{\Sigma f| D_M |}{n} = \frac{1582.2}{100} = 15.82$

Coefficient of Mean Deviation from Median $(MD_M) = \frac{MDM}{M} = \frac{15.82}{52.78} = .30$

8.10.4 MERITS OF MEAN DEVIATION

1. We can calculate mean deviation very easily.
2. Mean deviation is based on all the items of the Data. Change in any value of the data is also going to affect mean deviation.
3. As it is based on all the items of the data, it is not affected by the extreme values of the data.
4. Mean deviation can be calculated from Mean, Median or Mode.
5. Mean deviation is a rigidly defined method of measuring dispersion. 6. Mean deviation can be used for comparison of two different series.

8.10.5 LIMITATIONS OF MEAN DEVIATION

1. While calculating the mean deviation, we consider only positive sign and ignore the negative sign.
2. In case mean deviation is calculated from mode, it is not a reliable measure of dispersion as mode is not a true representative of the series.
3. It is very difficult to calculate Mean Deviation in case of open ended series.
4. Mean deviation is not much capable of further statistical calculations.
5. In case we have Mean Deviation of two different series, we cannot calculate combined mean deviation of the data.
6. In case value of Mean, Median or Mode is in fraction, it is difficult to calculate mean deviation.

8.11 TEST YOUR UNDERSTANDING (C)

1. Calculate Mean Deviation from i) Mean, ii) Median, iii) Mode X: 7, 4, 10, 9, 15, 12, 7, 9, 7

2. With Median as base calculate Mean Deviation of two series and compare variability:

Series A:	3484	4572	4124	3682	5624	4388	3680	4308
Series B:	487	508	620	382	408	266	186	218

3. Calculate Co-efficient of mean deviation from Mean, Median and Mode from the following data

X:	4	6	8	10	12	14	16
f:	2	1	3	6	4	3	1

4. Calculate Co-efficient of Mean Deviation from Median.

X;	20-25	25-30	30-40	40-45	45-50	50-55	55-60	60-70	70-80
F:	7	13	16	28	12	9	7	6	2

5. Calculate M.D. from Mean and Median

X	0-10	10-20	20-30	30-40	40-50
f	6	28	51	11	4

6. Calculate Co-efficient of Mean Deviation from Median.

X;	16-20	21-25	26-30	31-35	36-40	41-45	46-50	51-55	56-60
F:	8	13	15	20	11	7	3	2	1

Answers

1. **2.35, 2.33, 2.56**
2. **11.6%, 30.73%**
3. **.239, .24, .24**
4. **.214**
5. **M.D. (Mean) 6.572, Coefficient of M.D. (Mean) .287, M.D. (Median) 6.4952, Coefficient of M.D. (Median) .281**
6. **.22**

8.12 MEASURE OF DISPERSION – STANDARD DEVIATION

Standard deviation is assumed as best method of calculating deviations. This method was given by great statistician Karl Pearson in the year 1893. In case of Mean deviation, when we take deviations from actual mean, the sum of deviations is always zero. In order to avoid this problem, we have to ignore the sign of the deviations. However, in case of Standard Deviation this problem is solved by taking the square of the deviations, because when we take a square of the negative sign, it is also converted into the positive sign. Then after calculating the Arithmetic mean of the deviations we again take square root, to find out standard deviation. In other words we can say that "Standard Deviation is the square root of the Arithmetic mean of the squares of deviation of the item from its Arithmetic mean."

The standard deviation is always calculated from the Arithmetic mean and is an absolute measure of finding the dispersion. We could also find a relative measure of standard deviation which is known as coefficient of standard deviation.

Coefficient of Standard Deviation –

Coefficient of Deviation is the relative measure of the standard deviation and can be calculated by dividing the Value of Standard Deviation with the Arithmetic Mean. The value of coefficient always lies between 0 and 1, where 0 indicates no Standard Deviation and 1 indicated 100% standard deviation. Following is the formula for calculating coefficient of Standard Deviation.

$$\text{Coefficient of Standard Deviation} = \frac{SD}{\bar{X}}$$

Coefficient of Variation –

Coefficient of Variation is also relative measure of the standard deviation, but unlike Coefficient of Standard Deviation it is not represented in fraction rather it is represented in terms of %age. It can be calculated by dividing the Value of Standard Deviation with the Arithmetic Mean and then multiplying resulting figure with 100. The value of coefficient always lies between 0 and 100. Following is the formula for calculating coefficient of Standard Deviation. Low Coefficient of Variation implies less variation, more uniformity and reliability. Contrary to this higher Coefficient of Variation implies more variation, less uniformity and reliability.

$$\text{Coefficient of Standard Deviation} = \frac{SD}{\bar{X}} \times 100$$

Variance –

Variance is the square of the Standard Deviation. In other words it is Arithmetic mean of square of Deviations taken from Actual Mean of the data. This term was first time used by R. A. Fischer in 1913. He used Variance in analysis of financial models. Mathematically:

$$\text{Variance} = (\text{Standard Deviation})^2 \text{ or } \sigma^2$$

8.12.1 STANDARD DEVIATION IN CASE OF INDIVIDUAL SERIES

Following are the formula for calculating Standard Deviation in case of the Individual Series:

1. **ACTUAL MEAN METHOD –** In this method we take deviations from actual mean of the data.

$$\text{Standard Deviation (SD or } \sigma) = \sqrt{\frac{\sum x^2}{n}}$$

Where $x = X - \bar{X}$

n = Number of Items.

2. **ASSUMED MEAN METHOD -** In this method we take deviations from assumed mean of the data. Any number can be taken as assumed mean, however for sake of simplicity it is better to take whole number as assumed mean.

Standard Deviation (SD or σ) $= \sqrt{\frac{\sum dx^2}{n} - \left(\frac{\sum dx}{n}\right)^2}$ Where $dx = X - A$ n = Number of Items.

3. **DIRECT METHODS -** In this method we don't take deviations and standard deviation is calculated directly from the data.

Standard Deviation (SD or σ) $= \sqrt{\frac{\sum X^2}{n} - \left(\frac{\sum X}{n}\right)^2}$

Example 10. Following are the marks obtained by Students of a class in a test. Calculate Standard Deviation using (i) Actual Mean (ii) Assumed Mean (iii) Direct Method. Also calculate Coefficient of Standard Deviation.

Marks	5	7	11	16	15	12	18	12

Solution:

1. Standard Deviation using Actual Mean

Marks X	$x = X - \bar{X}$ **(Where $\bar{X}$ = 12)**	x^2
5	-7	49
7	-5	25
11	-1	01
16	4	16
15	3	09
12	0	00
18	6	36
12	0	00
$\sum X = 96$		$\sum x^2 = 136$

Mean $(\bar{X}) = \frac{\sum X}{n} = \frac{96}{8} = 12$

Standard Deviation (SD or σ) $= \sqrt{\frac{\sum x^2}{n}} = \sqrt{\frac{136}{8}} = \sqrt{17} = 4.12$

Coefficient of Standard Deviation $= \frac{SD}{\bar{X}} = \frac{4.12}{12} = .34$

2. Standard Deviation using Assumed Mean

Marks X	dx **= X -** A **(Where** A **= 11)**	dx^2
5	-6	36
7	-4	16
11	0	00
16	5	25
15	4	16
12	1	01
18	7	49
12	1	01
$\sum$**X = 96**	$\sum dx$ **= 8**	$\sum dx^2$ **= 144**

Mean $(\bar{X}) = A + \frac{\sum dx}{n} = 11 + \frac{8}{8} = 12$

Standard Deviation $(\sigma) = \sqrt{\frac{\sum dx^2}{n} - \left(\frac{\sum dx}{n}\right)^2} = \sqrt{\frac{144}{8} - \left(\frac{8}{8}\right)^2} = \sqrt{18 - 1} = \sqrt{17} = 4.12$

Coefficient of Standard Deviation $= \frac{SD}{\bar{X}} = \frac{4.12}{12} = .34$

3. Standard Deviation by Direct Method

Marks X	**X** ^{2}
5	25
7	49
11	121
16	256
15	225
12	144
18	324
12	144
$\sum$**X = 96**	$\sum$X2 **= 1288**

Mean $(\bar{X}) = \frac{\sum X}{n} = \frac{96}{8} = 12$

Standard Deviation $(\sigma) = \sqrt{\frac{\sum X^2}{n} - \left(\frac{\sum X}{n}\right)^2} = \sqrt{\frac{1288}{8} - \left(\frac{96}{8}\right)^2} = \sqrt{161 - 144} = \sqrt{17} = 4.12$

Coefficient of Standard Deviation $= \frac{SD}{\bar{X}} = \frac{4.12}{12} = .34$

Example 11. Two Players scored following scores in 10 cricket matches. On base of their performance find out which is better scorer and also find out which player is more consistent.

Player X	26	24	28	30	35	40	25	30	45	17
Player Y	10	15	24	26	34	45	25	31	20	40

Solution:

Mean and Standard Deviation of Player X

Score X	x = **X** - $\bar{X}$ **(Where** $\bar{X}$ **= 30)**	x^2
26	-4	16
24	-6	36
28	-2	2
30	0	0
35	5	25
40	10	100
25	-5	25
30	0	0
45	15	225
17	-13	169
$\sum$X = 300		$\sum x^2$ **= 600**

Mean $(\bar{X}) = \frac{\Sigma K}{n} = \frac{300}{10} = 30$

Standard Deviation (SD or σ) $= \sqrt{\frac{\Sigma x^2}{n}} = \sqrt{\frac{600}{10}} = \sqrt{60} = 7.746$

Coefficient of Variation $= \frac{SD}{\bar{X}} \times 100 = \frac{7.746}{30} \times 100 =$ 25.82%

Mean and Standard Deviation of Player Y

Score Y	y = **Y** -Y $^-$ **(Where** $\bar{Y}$ **= 27)**	y^2
10	-17	289
15	-12	144
24	-3	9
26	-1	1
34	7	49
45	18	324

25	-2	4
31	4	16
20	-7	49
40	13	169
$\sum X = 270$		**$\sum x^2 = 1054$**

Mean $(\bar{Y}) = \frac{\sum F}{n} = \frac{270}{10} = 27$

Standard Deviation (SD or σ) $= \sqrt{\frac{\sum y^2}{n}} = \sqrt{\frac{1054}{10}} = \sqrt{105.40} = 10.27$

Coefficient of Variation $= \frac{SD}{\bar{Y}} \times 100 = \frac{10.27}{27} \times 100 = 38.02\%$

Conclusion:

1. As average score of Player X is more than Player Y, he is better scorer. 2. As Coefficient of Variation of Player X is less than Player Y, he is more consistent also.

8.12.2 STANDARD DEVIATION IN CASE OF DISCRETE SERIES

Following are the formula for calculating Standard Deviation in case of the Discrete Series: 1. **Actual Mean Method –** In this method we take deviations from actual mean of the data.

Standard Deviation (SD or σ **)** $= \sqrt{\frac{\sum f x^2}{n}}$

Where x = X - $\bar{X}$

f = Frequency

n = Numberof Items.

2. **Assumed Mean Method -** In this method we take deviations from assumed mean of the data.

Standard Deviation (SD or σ **)** $= \sqrt{\frac{\sum f dx^2}{n} - \left(\frac{\sum f dx}{n}\right)^2}$

Wheredx = X – A

n = Numberof Items.

3. **Direct Methods -** In this method we don’t take deviations and standard deviation is calculated directly from the data.

Standard Deviation (SD or σ **)** $= \sqrt{\frac{\sum f X^2}{n} - \left(\frac{\sum f X}{n}\right)^2}$

Example 12. Following are the marks obtained by Students of a class in a test. Calculate Standard Deviation using (i) Actual Mean (ii) Assumed Mean (iii) Direct Method.

Marks	5	10	15	20	25	30	35
Frequency	2	7	11	15	10	4	1

Solution:

1. Standard Deviation using Actual Mean

Marks X	**f**	**fX**	$x = X - \bar{X}$ **($\bar{X}$ = 19)**	x^2	fx^2
5	2	10	-14	196	392
10	7	70	-9	81	567
15	11	165	-4	16	176
20	15	300	1	1	15
25	10	250	6	36	360
30	4	120	11	121	484
35	1	35	16	256	256
	N = 50	**∑fX = 950**			**$\sum x^2$ = 2250**

Mean $(\bar{X}) = \frac{\sum fX}{n} = \frac{950}{50} = 19$

Standard Deviation (SD or σ) = $\sqrt{\frac{\sum fx^2}{n}} = \sqrt{\frac{2250}{50}} = \sqrt{45} = 6.708$

2. Standard Deviation using Assumed Mean

Marks X	**f**	dx **= X - A (A = 20)**	dx^2	**fdx**	fdx^2
5	2	-15	225	-30	450
10	7	-10	100	-70	700
15	11	-5	25	-55	275
20	15	0	0	0	0
25	10	5	25	50	250
30	4	10	100	40	400
35	1	15	225	15	225
	N = 50			**∑fdx = -50**	**$\sum fdx^2$ = 2300**

$$= \sqrt{\frac{\sum fdx^2}{n} - \left(\frac{\sum fdx}{n}\right)^2}$$

$$= \sqrt{\frac{2300}{50} - \left(\frac{-50}{50}\right)^2} = \sqrt{46 - 1} = \sqrt{45} = 6.708$$

Standard Deviation (σ)

3. Standard Deviation using Direct Method

Marks X	f	X^2	fX	fX^2
5	2	25	10	125
10	7	70	70	700
15	11	225	165	2475
20	15	400	300	6000
25	10	625	250	6250
30	4	900	120	3600
35	1	1225	35	1225
	N = 50		**$\sum fX = 950$**	**$\sum fX^2 = 20300$**

Standard Deviation (σ) $= \sqrt{\frac{\sum fX^2}{n} - \left(\frac{\sum fX}{n}\right)^2}$

$= \sqrt{\frac{20300}{50} - \left(\frac{950}{50}\right)^2} = \sqrt{406 - 361} = \sqrt{45} = 6.708$

8.12.3 STANDARD DEVIATION IN CASE OF CONTINUOUS SERIES

In case of continuous series the calculation will remain same as in case of discrete series but the only difference is that instead of taking deviations from data, deviations are taken from Mid value of the data. Formulas are same as discussed above for discrete series.

Example 13. Following are the marks obtained by Students of a class in a test. Calculate Standard Deviation using (i) Actual Mean (ii) Assumed Mean (iii) Direct Method. Also calculate coefficient of variation and Variance.

Marks	5-10	10-15	15-20	20-25	25-30	30-35	35-40
Frequency	2	7	11	15	10	4	1

Solution:

1. Standard Deviation using Actual Mean

Marks X	m	f	fX	$x = m - \bar{X}$ ($\bar{X} = 21.5$)	x^2	fx^2
5-10	7.5	2	15	-14	196	392
10-15	12.5	7	87.5	-9	81	567
15-20	17.5	11	192.5	-4	16	176
20-25	22.5	15	337.5	1	1	15

25-30	27.5	10	275	6	36	360
30-35	32.5	4	130	11	121	484
35-40	37.5	1	37.5	16	256	256
		N = 50	**∑fX = 1075**			∑x² **= 2250**

Mean $(\bar{X}) = \frac{\sum fK}{n} = \frac{1075}{50} = 21.5$

Standard Deviation (SD or σ) = $\sqrt{\frac{\sum fx^2}{n}} = \sqrt{\frac{2250}{50}} = \sqrt{45} = 6.708$

2. Standard Deviation using Assumed Mean

Marks X	**m**	**f**	dx **= X - A (A = 22.5)**	dx^2	**Fdx**	fdx^2
5-10	7.5	2	-15	225	-30	450
10-15	12.5	7	-10	100	-70	700
15-20	17.5	11	-5	25	-55	275
20-25	22.5	15	0	0	0	0
25-30	27.5	10	5	25	50	250
30-35	32.5	4	10	100	40	400
35-40	37.5	1	15	225	15	225
		N = 50			**∑fdx = -50**	$\sum fdx^2$ **= 2300**

$$= \sqrt{\frac{\sum fdx^2}{n} - \left(\frac{\sum fdx}{n}\right)^2}$$

$$= \sqrt{\frac{2300}{50} - \left(\frac{-50}{50}\right)^2} = \sqrt{46 - 1} = \sqrt{45} = 6.708$$

Standard Deviation (σ)

3. Standard Deviation using Direct Method

Marks X	**m**	**f**	X2	**fX**	fX^2
5-10	7.5	2	56.25	15	112.5
10-15	12.5	7	156.25	87.5	1093.75
15-20	17.5	11	306.25	192.5	3368.75
20-25	22.5	15	506.25	337.5	7593.75
25-30	27.5	10	756.25	275	7562.5
30-35	32.5	4	1056.25	130	4225
35-40	37.5	1	1406.25	37.5	1406.25

		N = 50		$\sum$fX = 1075	$\sum f X^2$ = 25366.5

Standard Deviation (σ) $=\sqrt{\frac{\sum fX^2}{n} - \left(\frac{\sum fX}{n}\right)^2}$

$=\sqrt{\frac{25366.5}{50} - \left(\frac{1075}{50}\right)^2} = \sqrt{507.25 - 462.25} = \sqrt{45} = 6.708$

Coefficient of Standard Deviation $= \frac{SD}{\bar{X}} \times 100 = \frac{6.708}{21.5} \times 100$ =31.2%

Variance =(Standard Deviation)^2 or σ^2 = $(6.708)^2$ = 45

8.12.4 COMBINED STANDARD DEVIATION

The main benefit of standard deviation is that if we know the mean and standard deviation of two or more series, we can calculate combined standard deviation of all the series. This feature is not available in other measures of dispersion. That's why we assume that standard deviation is best measure of finding the dispersion. Following formula is used for this purpose:

$$\sigma_{123} = \sqrt{\frac{n_1 \sigma_1^2 + n_2 \sigma_2^2 + n_3 \sigma_3^2 + n_1 d_1^2 + n_2 d_2^2 + n_3 d_3^2}{n_1 + n_2 + n_3}}$$

Where

n_1, n_2, n_3 = number of items in series 1, 2 and 3 σ_1, σ_2, σ_3 = standard deviation of series 1, 2 and 3 d_1, d_2, d_3 = difference between mean of the series and combined mean for 1, 2 and 3. ***Example14. Find the combined standard deviaiton for the following data***

	***Firm** A*	***Firm** B*
No. of Wage Workers	70	60
***Average Daily Wage** (Rs.)*	40	35
S.D of wages	***8***	***10***

Solution : Combined mean wage of all the workers in the two firms will be

$$\overline{X_{12}} = \frac{N_1\overline{X_1}+N_2\overline{X_2}}{N_1+N_2}$$

Where N_1 = Number of workers in Firm A

N_2 = Number of workers in Firm B

$\overline{X_1}$ = Mean wage of workers in Firm A

and $\overline{X_2}$ = Mean wage of workers in Firm B

We are given that

$N_1 = 70 \qquad N_2 = 60$

$\overline{X_1} = 40 \qquad \overline{X_2} = 35$

∴ Combined Mean, $\overline{X_{12}}$

$= \frac{(70 \times 40)+(60 \times 35)}{70+60}$

$= \frac{4900}{130}$

$= Rs. 37.69$

Combined Standard Deviation =

$$\sigma_{123} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1 d_1^2 + n_2 d_2^2}{n_1 + n_2}}$$

$d_1 = 40 - 37.69 = 2.31$

$d_2 = 35 - 37.69 = -2.69$

$$\sigma_{123} = \sqrt{\frac{70\,(8)^2 + 60\,(10)^2 + 70\,(2.31)^2 + 60\,(-2.69)^2}{70+60}} = 9.318$$

Example 15. Find the missing values

	Firm A	*Firm B*	*Firm C*	*Combined*
No. of Wage Workers	50	***?***	90	**200**
Average Daily Wage (Rs.)	113	?	115	**116**
S.D of wages	***6***	***7***	***?***	***7.746***

Solution :

Combined n = $n_1 + n_2 + n_3$

$200 = 50 + n_2 + 90$ N_2

= 60

Now Combined mean wage of all the workers in the two firms will be

$$\overline{X_{12}} = \frac{N_1\overline{X_1} + N_2\overline{X_2} + N_3\overline{X_3}}{N_1+N_2+N_3}$$

We are given that

$N_1 = 50 \; N_2 = 60 \; N_3 = 90 \qquad X_1 = 113 \; X_2$

$X_3 = 115 \qquad \overline{X_{123}} = 116$

=?

∴ Combined Mean, $\overline{X_{12}}$

$$116 = \frac{(50 \times 113)+(60 \times \overline{X_2}) + (90 \times 115)}{50+60+90}$$

$$116 = \frac{565 + (60 \times \overline{X_2}) + 1035}{50 + 60 + 90}$$

$$2320 = 1600 + 6\,\overline{X_2}$$

$$\overline{X_2} = 120$$

Combined Standard Deviation =

$$\sigma_{123} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + {}_3\sigma_3^2 + n_1 d_1^2 + n_2 d_2^2 + n_3 d_3^2}{n_1 + n_2 + n_3}}$$

$d_1 = 113 – 116 = -3$
$d_2 = 120 – 116 = 4$ d_3
$= 115 – 116 = -1$

$$\sigma_{123} = \sqrt{\frac{50\,(6)^2 + 60\,(7)^2 + 90\,(\sigma_3)^2 + 50\,(-3)^2 + 60\,(4)^2 + 90\,(-1)^2}{50 + 60 + 90}} = 7.746$$

Squaring the both sides

$$60 = \frac{180+294+9\sigma_3^2 45+96+9}{200}$$

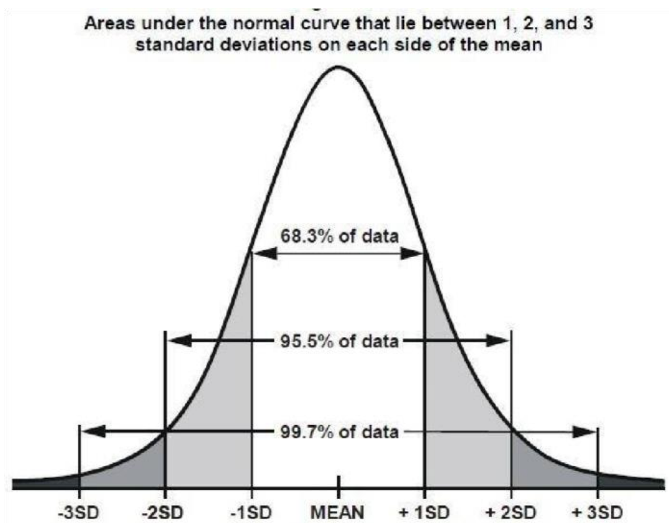
$1200 = 9\,\sigma_3{}^2 +$
624 $\sigma_3 = 8$

8.12.5 Properties of Standard Deviation

1. Standard Deviation of first 'n' natural numbers is $\sqrt{\frac{n^2 - 1}{12}}$
2. It is independent of change in origin it means it is not affected even if some constant is added or subtracted from all the values of the data.
3. It is not independent of change in scale. So if we divide or multiply all the values of the data with some constant, Standard Deviation is also multiplied or divided by same constant.
4. We can calculate combined Standard Deviation by following formula:

$$\sigma_{123} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_3\sigma_3^2 + n_1 d_1^2 + n_2 d_2^2 + n_3 d_3^2}{n_1 + n_2 + n_3}}$$

5. In case of normal distribution following results are found:



68.27% item lies within the range of : $\overline{X} \pm \sigma$95.45% item lies within the range of :$\overline{X} \pm 2\,\sigma$

99.73% item lies within the range of :$\overline{X} \pm 3\sigma$

6. In case of normal distribution there is relation between Quartile Deviation, Mean Deviation and Standard Deviation which is as follows:

   6 (Q.D.) = 5 (M.D.) = 4 (S.D.)

7. In perfect symmetric distribution following result follows: Range = 6 (S.D.)
8. When we take square of Standard Deviation it is called Variance.

   Variance = $(S.D)^2$

8.12.6 MERITS OF STANDARD DEVIATION

1. It is rigidly defined.
2. It is best measure to find out deviations.
3. It is based on arithmetic mean.
4. It is based on all the values.
5. We can find combined standard deviation of different series under this.
6. It is capable of further algebraic treatment. 7. By finding coefficient of variation we can compare two different series.

8.12.7 Limitations of Standard Deviation 1. It is comparatively difficult to calculate.

2. It is mostly affected by extreme values. 3. Common people are not aware about the concept of standard deviation.

8.13 TEST YOUR UNDERSTANDING (D)

1. Calculate Standard Deviation and find Variance:

X:	5	7	11	16	15	12	18	12

2. Two Batsmen X and Y score following runs in ten matches. Find who is better Scorer and who is more consistent?

X:	26	24	28	30	35	40	25	30	45	17
Y:	10	15	24	26	34	45	25	31	20	40

3. Calculate S.D, coefficient of SD, coefficient of Variation:

X	15	25	35	45	55	65
f	2	4	8	20	12	4

4. Find Standard Deviation.

X;	5-10	10-15	15-20	20-25	25-30	30-35
F:	2	9	29	24	11	6

5. Find Standard Deviation and coefficient of variation.

X;	30-39	40-49	50-59	60-69	70-79	80-89	90-99
F:	1	4	14	20	22	12	2

6. Find Standard Deviation.

X; 0-50 50-100 100-200 200-300 300-400 400-600

F: 4 8 10 15 9 7

7. Find combined Mean and Combined Standard Deviation:

Part	No. of Items	Mean	S.D.
1	200	25	3
2	250	10	4
3	300	15	5

8. Find missing information:

	Group I	Group II	Group III	Combined
No. of Items	200	?	300	750
Mean	?	10	15	16
S.D	3	4	?	7.1924

Answers

1. 4.12, 16.97

2. **X is better and consistent,X mean 30 CV 25.82%, Y mean 27 CV 38.02%**
3. **11.83, 0.265, 26.5%**
4. **5.74**
5. **12.505, 18.36%**
6. **141.88**
7. **16, 7.2**
8. **250, 25, 5**

8.14 LET US SUM UP

- **Dispersion shows that whether average is a good representative of the series or not.**
- **High dispersion mean values differ more than its average.**
- **There are two measures of dispersion, Absolute measure and relative measure.**
- **There are four methods that can be used for measuring the dispersion namely, Range, Quartile Deviation, Mean Deviation and Dispersion.**
- **Range is simples method of dispersion.**
- **Mean deviation can be calculated from Mean, Median or Mode**
- **Standard Deviation is the best measure of Dispersion.**
- **If we know standard deviation of two series we can calculate combined standard deviation.**

8.15 KEY TERMS

- **DISPERSION:**Dispersion shows the extent to which individual items in the data differs from its average. It is a measure of difference between data and the individual items. It indicates that how that are lacks the uniformity.
- **RANGE:** Range as the difference between highest value of the data and the lowest value of the data. The more is the difference between highest and the lowest value, more is the value of Range which shows high dispersion.
- **QUARTILE DEVIATION:** Quartile deviation is the Arithmetic mean of the difference between Third Quartile and the First Quartile of the data.
- **MEAN DEVIATION** . Mean Deviation is the value obtained by taking arithmetic mean of the deviations obtained by deducting average of data whether Mean, Median or Mode from values of data, ignoring the signs of the deviations.

- **STANDARD DEVIATION:** Standard Deviation is the square root of the Arithmetic mean of the squares of deviation of the item from its Arithmetic mean.
- **VARIANCE:** It is square of Standard Deviation.
- **ABSOLUTE MEASURE:**Absolute measure of dispersion is one which is expressed in the same statistical unit in which the original values of that data are expressed. For example if original data is represented in kilograms, the dispersion will also be represented in kilogram.
- **RELATIVE MEASURE:**The relative measure of dispersion is independent of unit of measurement and is expressed in pure number. Normally it is a ratio of the dispersion to the average of the data.
- **COEFFICIENT OF STANDARD DEVIATION:**Coefficient of Deviation is the relative measure of the standard deviation and can be calculated by dividing the Value of Standard Deviation with the Arithmetic Mean. The value of coefficient always lies between 0 and 1, where 0 indicates no Standard Deviation and 1 indicated 100% standard deviation.

8.16 REVIEW QUESTIONS

1. What is Dispersion. What are uses of measuring Dispersion.
2. What are features of good measure of Dispersion.
3. What are absolute and relative measure of dispersion.
4. What is range? Give its merits and limitations.
5. What is Quartile deviations. Give its merits and limitations.
6. What is mean deviation. How it is calculated. Give its merits and limitations.
7. What is standard deviation? How it is calculated. Give its merits and limitations.
8. How combined standard deviation can be calculated.
9. Give properties of standard deviation.

8.19 FURTHER READINGS

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2. S.C. Gupta, *Fundamentals of Statistics*, Himalaya Publishing House.
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B. COM (DIGITAL)

SEMESTER II

COURSE: BUSINESS MATHEMATICS AND STATISTICS

UNIT 9 – CORRELATION

STRUCTURE

9.0 OBJECTIVES

After studying the Unit, students will be able to

- Define what is Correlation.
- Distinguish between different types of correlation.
- Understand benefits of correlation.
- Find correlation using the graphic method.
- Calculate correlation by Karl Pearson Method.
- Measure correlation using Rank correlation method.

9.1 INTRODUCTION

When we study measurement of central tendency, dispersion analysis, skewness analysis etc., we study the nature and features of data in which only one variable is involved. However, In our daily life we come across a number of things in which two or more variables are involved and such variables may be related to each other. As these variables are related to each other, it is important to understand the nature of such relation and the extent of such relation. Identification of such relation helps us in solving a number of problems of dalily life. This is not only helpful in our daily life but also helpful in the solving many business problems. For example if a businessman knows the relation between income and demand, Price and Demand, etc, it will help him in formulation of business plans. Correlation is one such statistical technique that helps us in understanding relation between two or more variables.

9.2 MEANING OF CORRELATION:

Correlation is a statistical technique which studies the relation between two or more variables. It studies that how to variables are related to each other. It studies that how change in value of one variable affects the other variable, for example in our daily life we will find the relation between income and expenditure, income and demand, Price and Demand age of husband and wife etcetera correlation helps in understanding such relations of different variables two variables are said to be related to each other when change in value of one variable so results in to change in value of other variable.

According to Croxton and Cowden, "When the relationship is of a quantitative nature, the appropriate statistical tool for discovering and measuring the relationship and expressing it in a brief formula is known as correlation."

According to W.I. King, "Correlation means that between two series or groups of data there exists some casual connection."

9.3 USES OF CORRELATIONS:

1. It helps us in understanding the extent and direction of relation between two variables. It shows, whether two variables are positively correlated or negatively correlated. It also shows that whether relation between two variables is high or low.
2. Correlation also helps in prediction of future, for example if we know relation between monsoon and agricultural produce, we can predict that what will be the level of produce on basis of monsoon prediction. We can also predict price of Agricultural Products depending upon level of produce.
3. With the help of correlation, we can find the value of one variable when the value of other variable is known. This can be done by using the statistical technique called regression analysis.
4. Correlation also helps in business and Commerce. Businessman can fix price of its product using the correlation analysis. Correlation also helps him in deciding business policy.
5. Correlation also helps government in deciding its economic policy. With the help of correlation government can study relation of various economic variable, thus government can decide their economic policies accordingly.
6. Correlation is also helpful in various statistical Analysis. There are many Statistical techniques that use correlation for further analysis.

9.4 TYPES OF CORRELATION

9.4.1 POSITIVE, NEGATIVE AND NO CORRELATION

a. POSITIVE CORRELATION: It is a situation in which two variables move in the same direction. In this case if the value of one variable increase the value of other variable also increase. Similarly, if the value of one variable decrease, the value of other variable also decrease. So, when both the variables either increase or decrease, it is known as positive correlation. For example we can find Positive correlation between Income and Expenditure, Population and Demand of food products, Incomes and Savings etc. Following data shows positive correlation between two variables:

Height of Persons	158	161	164	166	169	172	17

: X							4
Weight of Person : Y	61	63	64	66	67	69	72

b. NEGATIVE OR INVERSE CORRELATION: When two variables move in opposite direction from each other, it is known as negative or inverse correlation. In other words, we can say that when the value of one variable increase value of other variable decrease, it is called negative correlation. In our life we find negative correlation between a number of variables, for example there is negative correlation between Price and Demand, Number of Workers and Time required to complete the work etc. Following data shows the negative correlation between two variables:

Price of Product : X	1	2	3	4	5
Demand of Product : Y	50	45	40	35	30

c. Zero or No Correlation: When two variables does not show any relation, it is known as zero or no correlation. In other words, we can say that in case of zero correlation, the change in value of one variable does not affect the value of other variable. In this case two variables are independent from each other. For example, there is zero correlation between height of the student and marks obtained by the student.

9.4.2 SIMPLE AND MULTIPLE CORRELATION:

a. **SIMPLE CORRELATION**: When we study relation between two variables only, it is known as simple correlation. For example, relation between income and expenditure, Price and Demand, are situations of simple correlation.

b. **MULTIPLE CORRELATION**: Multiple correlation is a situation in which more than two variables are involved. Here relation between more than two variables are studied together, for example if we are studying the relation between income of the consumer, price if the product and demand of the product, it is a situation of multiple correlation.

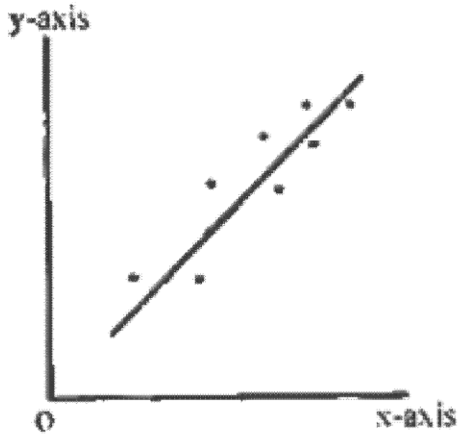
9.4.3 TOTAL AND PARTIAL CORRELATION:

a. **TOTAL CORRELATION**: In case we study relation of more than two variables and all the variables are taken together, it is a situation of total correlation. For example, if we are studying the relation between the income of the consumer, price of the product and demand of the product, taking all the factors together it is called total correlation.
b. **PARTIAL CORRELATION**: In case of partial correlation more than two variables are involved, but while studying the correlation we take only two factors in consideration assuming that the value of other factors is constant. For example, while studying the relation between income of the consumer, price of the product and demand of the product, we take into consideration only relation between price of the product and demand of the product assuming that income of the consumer is constant.

9.4.4 LINEAR AND NON LINEAR CORRELATION:

a. LINEAR CORRELATION: When the change in value of one variable results into constant ratio of change in the value of other variable, it is called linear correlation. In such case if we draw the values of two variables on the graph paper, all the points on the graph paper will fall on a straight line. For example, every change in income of consumer by Rs. 1000 results into increase in consumption by 10 k.g., is known as linear correlation. Following data shows example of linear correlation:

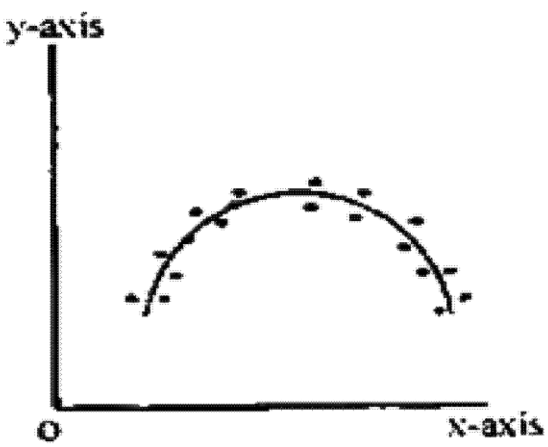
Price of Product : X	1	2	3	4	5
Demand of Product : Y	50	45	40	35	30



Linear Correlation

b. NON - LINEAR CORRELATION: When the change in value of one variable does not result into constant ratio of change in the value of other variable, it is called non linear correlation. In such case, if we draw the value of two variables on the graph paper all the points will not fall in the straight line on the graph. Following data shows non linear correlation between two variables:

Price of Product : X	1	2	3	4	5
Demand of Product : Y	50	40	35	32	30



Non Linear Correlation

9.5 SCATTER DIAGRAM METHOD

Scatter Diagram is one of the oldest and simple method of measuring the correlation. This is a graphic method of measuring the correlation. This method use diagram representation of bivariate data to find out degree and direction of correlation. Under this method, values of the data are plotted on a graph paper by taking one variable on the x-axis and other variable on the y-axis. Normally independent variable is shown on x-axis whereas the value of the dependent variable is taken on the y-axis. Once all the values are drawn on the graph paper we can find out degree of correlation between two variables by looking at direction of dots on the graph. Scatter Diagram shows whether two variables are co related to each other or not. It also shows the direction of correlation whether positive or negative and the shows extent of correlation whether high or low. Following situations are possible in the scatter diagram.

1. PERFECT POSITIVE CORRELATION: After we plot two variables on the graph, if the points of graph fall in a straight line which moves from lower left hand side to the upper corner on the right hand side, then it is assumed that there is perfect positive correlation between the variables.

2. PERFECT NEGATIVE CORRELATION: After drawing the variables on the graph, if all the points fall in a straight line but direction of the points is downward from right hand corner to left hand side corner, then it is assumed that there is perfect negative correlation between the variable.

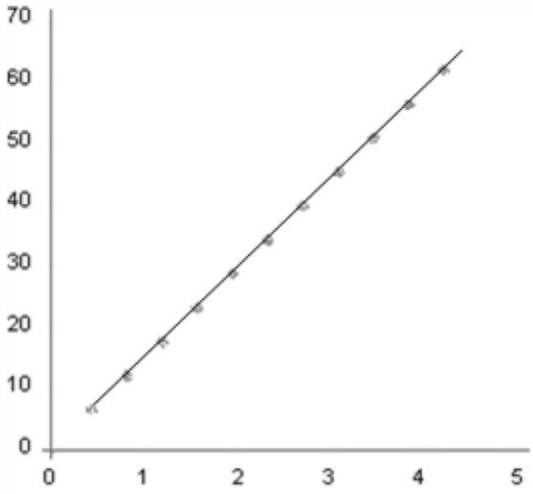
3. HIGH DEGREE OF POSITIVE CORRELATION: If we draw two variables on the graph and we find that the points move in upward direction from left hand corner to the right hand corner but not in a straight line, rather these are in narrow band, we can assume that there is high degree of positive correlation between the variables.

4. HIGH DEGREE OF NEGATIVE CORRELATION: After plotting the dots on a graph, if we find that all the dots move downward from left hand corner to the right hand side corner but not in a straight line rather in a narrow band, we can say that there is high degree of negative correlation between the variables.

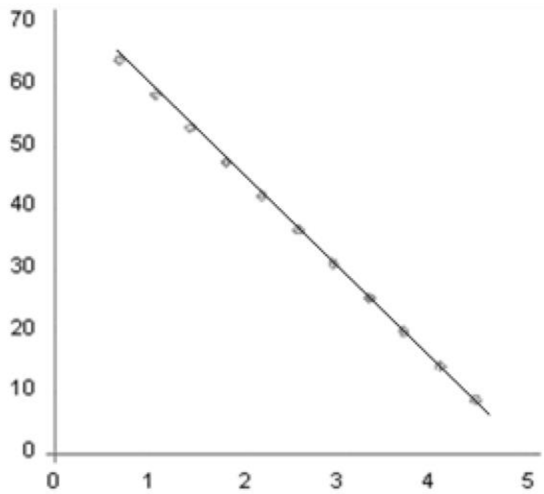
5. LOW DEGREE OF POSITIVE CORRELATION: In case the dots drawn on a graph paper moves upward from left side to right side but the dots are widely scattered, it can be said that there is low degree of positive correlation between the variables.

6. LOW DEGREE OF NEGATIVE CORRELATION: In case the points drawn on a graph are in downward direction from left side to right side but the points are widely scattered, it is the situation of low degree of negative correlation between the variables.

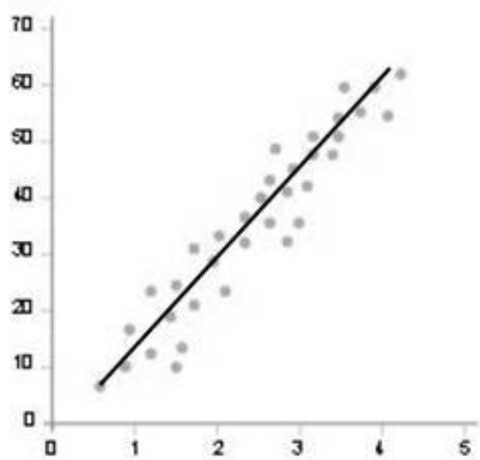
7. ZERO OR NO CORRELATION: Sometime find that the dots drawn on a graph paper does not move in any direction and are widely scattered in the graph paper, we can assume that there is no correlation between the two variables.



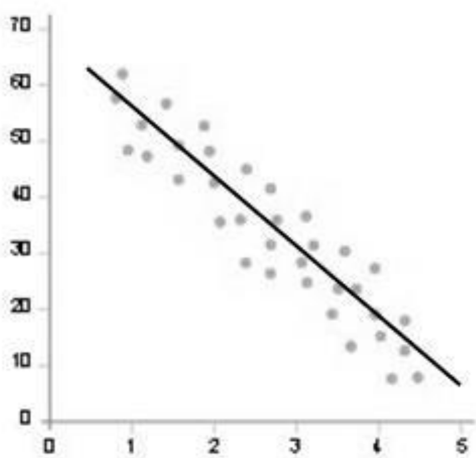
Perfect Positive Correlation



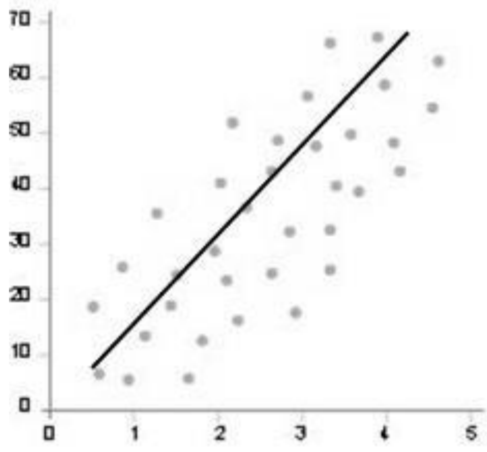
Perfect Negative Correlation



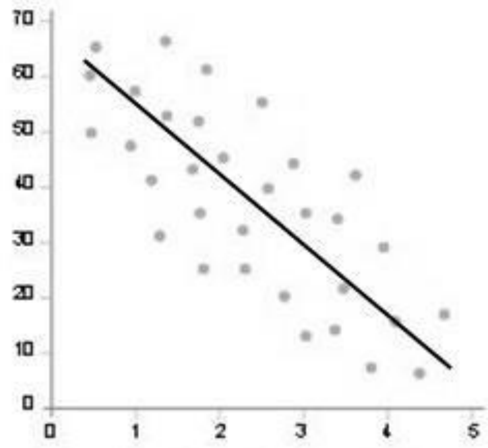
High Positive Correlation



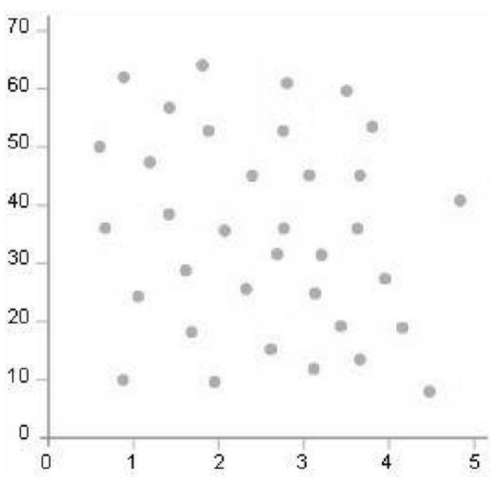
High Negative Correlation



Low Positive Correlation



Low Negative Correlation



No Correlation

9.6 KARL PEARSONS'S COEFFICIENT OF CORRELATION

Karl Peason's Coefficient of Correlation is the most important method of measuring the correlation. He was the first person who introduced the mathematical model of finding the correlation. Karl Peason's Coefficient of correlation is also denoted as 'Product Moment Correlation' also. The coefficient of correlation given by Karl Pearson is denoted as a symbol 'r'. It is the relative measure of finding the correlation. According to Karl Pearson we can determine correlation by dividing the product of deviations taken from mean of the data .

9.6.1 FEATURES OF KARL PEARSON'S COEFFICIENT OF CORRELATION

1. Karl Pearson's method is algebraic method of finding correlation.
2. The coefficient of correlation may be positive or negative.
3. This method is based on Arithmetic mean of the data and the standard deviation of the data.

4. The value of coefficient of correlation always lies between -1 and + 1. -1 refers to 100% negative correlation, plus one refers to 100% positive correlation, and zero refers to no correlation between the items.
5. This method is based on all the items of the Data.

9.6.2 DIRECT METHOD OF CALCULATING CORRELATION

Correlation can be calculated using the direct method without taking any mean. Following are the steps:

1. Take two series X and Y.
2. Find the sum of these two series denoted as ∑X and ∑Y.
3. Take the square of all the values of the series X and series Y.
4. Find the sum of the square so calculated denoted by $\sum X^2$ and $\sum Y^2$.
5. Multiply the corresponding values of series X and Y and find the product.
6. Sum up the product so calculated denoted by ∑ X Y.
7. Apply the following formula for calculating the correlation.

Coefficient of Correlation, $r = \frac{N\sum XY - (\sum X)(\sum Y)}{\sqrt{N\sum X^2 - (\sum X)^2}\sqrt{N\sum Y^2 - (\sum Y)^2}}$

Example 1. ***Calculate the Karl Pearson's coefficient of correlation for the following data***

X	2	3	1	5	6	4
Y	4	5	3	4	6	2

Solution :

X	Y	X2	Y2	XY
2	4	4	16	8
3	5	9	25	15
1	3	1	9	3
5	4	25	16	20
6	6	36	36	36
4	2	16	4	8
$\sum X = 21$	$\sum Y = 24$	$\sum X^2 = 91$	$\sum Y^2 = 106$	$\sum XY = 90$

$N = 6$

Coefficient of Correlation, $r = \frac{N\sum KF-(\sum K)(\sum F)}{\sqrt{N\sum\ ^{2}-(\sum K)^{2}}\sqrt{N\sum F^{2}-(\sum F)^{2}}}$

$$= \frac{6\times 90-21\times 24}{\sqrt{6\times 91-(21)^{2}}\sqrt{6\times 106-(24)^{2}}}$$

$$= \frac{540-504}{\sqrt{546-441}\sqrt{636-576}}$$

$$= \frac{36}{\sqrt{105}\sqrt{60}}$$

$$= \frac{36}{10.246\times 7.7459}$$

$$= \frac{36}{79.31} = 0.4539$$

$\Rightarrow \qquad r = 0.4539$

9.6.3 ACTUAL MEAN METHOD OF CALCULATING CORRELATION

Under this Correlation is calculated by taking the deviations from actual mean of the data. Following are the steps:

1. Take two series X and Y.
2. Find the mean of both the series X and Y, denoted by$\overline{X}$ and $\overline{Y}$.
3. Take deviations of series X from it mean and it is denoted by ‘x’.
4. Take deviations of series Y from it mean and it is denoted by ‘y’.
5. Take square of deviation of series X denoted by x^2.
6. Sum up square of deviations of series X denoted by $\sum x^2$.
7. Take square of deviation of series Y denoted by y^2.
8. Sum up square of deviations of series Ydenoted by $\sum y^2$.
9. Find the product of x and y and it is denoted by xy.
10. Find the sum of ‘xy ‘it is denoted by $\sum xy$
11. Apply the following formula for calculating the correlation.

$$r = \frac{\sum xy}{\sqrt{\sum x^{2}}\sqrt{\sum y^{2}}} = \frac{\sum(X-\overline{X})(Y-\overline{Y})}{\sqrt{\sum(X-\overline{X})^{2}}\sqrt{(Y-\overline{Y})^{2}}}$$

Example 2. ***Calculate Karl Pearson’s coefficient of correlation***

X	50	50	55	60	65	65	65	60	60	50
Y	11	13	14	16	16	15	15	14	13	13

Solution : When deviations are taken from actual arithmetic mean, 'r' is given by

$$r = \frac{\sum xy}{\sqrt{\sum x^2}\sqrt{\sum y^2}} = \frac{\sum(\mathrm{K}-\overline{\mathrm{X}})(\mathrm{F}-\overline{\mathrm{Y}})}{\sqrt{\sum(\mathrm{K}-\overline{\mathrm{X}})^2}\sqrt{(\mathrm{F}-\overline{\mathrm{Y}})^2}}$$

Where $x = \mathrm{X} - \overline{\mathrm{X}}$ =Deviation from $A.M.$ of X series
$y = Y - \overline{\mathrm{Y}}$ =Deviation from $A.M.$ of Y series

X	Y	$x = (\mathrm{X} - \mathrm{X})$	x^2	$y = (Y - \mathrm{Y})$	y^2	xy
50	11	−8	64	−3	9	24
50	13	−8	64	−1	1	8
55	14	−3	9	0	0	0
60	16	2	4	2	4	4
65	16	7	49	2	4	14
65	15	7	49	1	1	7
65	15	7	49	1	1	7
60	14	2	4	0	0	0
60	13	2	4	−1	1	−2
50	13	−8	64	−1	1	8
$\sum \mathrm{X} = 580$	$\sum Y = 140$		$\sum x^2 = 360$		$\sum y^2 = 22$	$\sum xy = 70$

Here, $N = 10$

$A.M.$ of X series, $\overline{\mathrm{X}} = \frac{\sum \mathrm{K}}{N} = \frac{580}{10} = 58$

$A.M.$ of Y series, $\overline{\mathrm{Y}} = \frac{\sum \mathrm{F}}{N} = \frac{140}{10} = 14$

Coefficient of Correlation, $r = \frac{\sum xy}{\sqrt{\sum x^2}\sqrt{\sum y^2}} = \frac{70}{\sqrt{360 \times 22}} = \frac{70}{\sqrt{7920}} = 0.7866$

$\Rightarrow \quad r = 0.7866$

9.6.4 ASSUMED MEAN METHOD OF CALCULATING CORRELATION

Under this Correlation is calculated by taking the deviations from assumed mean of the data.

Following are the steps:

1. Take two series X and Y.
2. Take any value as assumed mean for series X .
3. Take deviations of series X from its assumed mean and it is denoted by ‘dx’.
4. Find sum of deviations denoted by ∑dx.
5. Take square of deviation of series X denoted by dx^2 6. Sum up square of deviations of series X denoted by ∑ dx^2.
7. Take any value as assumed mean for series Y .
8. Take deviations of series Y from its assumed mean and it is denoted by ‘dy’.
9. Find sum of deviations of series Y denoted by ∑dy.
10. Take square of deviation of series Y denoted by dy^2 11. Sum up square of deviations of series Y denoted by ∑ dy^2.
12. Find the product of dx and dy and it is denoted by dxdy.
13. Find the sum of ‘dxdy ‘it is denoted by ∑ dxdy
14. Apply the following formula for calculating the correlation.

$$r = \frac{N \sum dxdy - (\sum dx)(\sum dy)}{\sqrt{N \sum dx^2 - (\sum dx)^2}\sqrt{N \sum dy^2 - (\sum dy)^2}}$$

Example 3. Compute coefficient of correlation from the following figures

City	A	B	C	D	E	F	G
Population (in thousands)	78	25	16	14	38	61	30
Accident Rate (per million)	80	62	53	60	62	69	67

Solution : Here, $N = 7$

Coefficient of Correlation, r is given by

$$r = \frac{N \sum dxdy - (\sum dx)(\sum dy)}{\sqrt{N \sum dx^2 - (\sum dx)^2}\sqrt{N \sum dy^2 - (\sum dy)^2}}$$

Where $dx =$ Deviations of terms of X series from assumed mean $A_K = X - A_K$

$dy =$ Deviations of terms of Y series from assumed mean $A_F = Y - A_F$

X	Y	$dx = X - A_K$ $A_K = 38$	$dy = Y - A_F$ $A_F = 67$	dx^2	dy^2	$dxdy$

70	80	32	13	1024	169	416
25	62	−13	−5	169	25	65
16	53	−22	−14	482	196	308
14	60	−24	−7	576	49	168
38	62	0	−5	0	25	0
61	69	23	2	529	4	46
30	67	−8	0	64	0	0
		$\sum dx = -12$	$\sum dy = -16$	$\sum dx^2 = 2846$	$\sum dy^2 = 468$	$\sum dxdy = 1003$

Here, $N = 7$

∴ Coefficient of Correlation, $r = \frac{7\times1003-(-12)(-16)}{\sqrt{7\times2846-(-12)^2}\sqrt{7\times468-(-16)^2}}$

$$= \frac{7021-192}{\sqrt{19{,}922-144}\sqrt{3276-256}}$$

$$= \frac{6829}{\sqrt{19{,}778}\sqrt{3020}} = 0.8837$$

⇒ $r = 0.8837$

9.6.5 STEP DEVIATION METHOD OF CALCULATING CORRELATION

Under this method assumed mean is taken but the difference is that after taking the deviation, these are divided by some common factor to get the step deviations. Following are the steps:

1. Take two series X and Y.
2. Take any value as assumed mean for series X .
3. Take deviations of series X from its assumed mean and it is denoted by ‘dx’. 4. Divide the value of ‘dx’ so obtained by some common factor to get dx' 5. Find sum of deviations denoted by $\sum dx'$.

6. Take square of deviation of series X denoted by dx'^2 7. Sum up square of deviations of series X denoted by $\sum dx'^2$.

8. Take any value as assumed mean for series Y .
9. Take deviations of series Y from its assumed mean and it is denoted by ‘dy’. 10. Divide the value of ‘dy’ so obtained by some common factor to get dy'
11. Find sum of deviations of series Y denoted by $\sum dy'$.
12. Take square of deviation of series Y denoted by dy'^2
13. Sum up square of deviations of series Y denoted by $\sum dy'^2$.
14. Find the product dx' of and dy' and it is denoted by $dx'd$ $'$.
15. Find the sum of ‘dxdy ‘it is denoted by $\sum dx'dy'$

16. Apply the following formula for calculating the correlation.

Coefficient of Correlation, $r = \frac{N\sum dx'dy' - (\sum dx')(\sum dy')}{\sqrt{N\sum dx'^2 - (\sum dx')^2}\sqrt{N\sum dy'^2 - (\sum dy')^2}}$

Example 4. ***Find the coefficient of correlation by Karl Pearson's method***

Price **($Rs.$)**	5	10	15	20	25
Demand **(kg)**	40	35	30	25	20

Solution :

X	Y	$dx = X - A$ $A = 15$	$dx' = \frac{dx}{C_1}$ $C_1 = 5$	$dy = Y - B$ $B = 30$	$dy' = \frac{dy}{C_1}$ $C_2 = 5$	dx'^2	d'^2	$dx'dy'$
5	40	-10	-2	10	2	4	4	-4
10	35	-5	-1	5	1	1	1	-1
15	30	0	0	0	0	0	0	0
20	25	5	1	-5	-1	1	1	-1
25	20	10	2	-10	-2	4	4	-4
			$\sum dx' = 0$		$\sum dy' = 0$	$\sum dx'^2 = 10$	$\sum dy'^2 = 10$	$\sum dx'dy' = -10$

Here $N = 5$

Coefficient of Correlation, $r = \frac{N\sum dx'dy' - (\sum dx')(\sum dy')}{\sqrt{N\sum dx'^2 - (\sum dx')^2}\sqrt{N\sum dy'^2 - (\sum dy')^2}}$

$$= \frac{5\times(-10) - 0\times 0}{\sqrt{5\times 10 - 0^2}\sqrt{5\times 10 - 0^2}}$$

$$= \frac{-50}{\sqrt{50}\times\sqrt{50}} = -1$$

$\Rightarrow$ $r = -1$

9.6.6 CALCULATING CORRELATION WITH HELP OF STANDARD DEVIATIONS

Under this method assumed mean is taken but the difference is that after taking the deviation, these are divided by some common factor to get the step deviations. Following are the steps:

1. Take two series X and Y.

2. Find the mean of both the series X and Y, denoted by $\overline{X}$ and $\overline{Y}$.
3. Take deviations of series X from it mean and it is denoted by ‘x’.
4. Take deviations of series Y from it mean and it is denoted by ‘y’.
5. Find the product of x and y and it is denoted by xy.
6. Find the sum of ‘xy ‘it is denoted by $\sum$ xy
7. Calculate the standard deviation of both series X and Y.
8. Apply the following formula for calculating the correlation.

$$r = \frac{\sum xy}{N\sigma_X\sigma_Y}$$

Example 5. ***Given***

No. of pairs of observations $= 10$

$\sum xy = 625$

X ***Series Standard Deviation*** $= 9$

Y ***Series Standard Deviation*** $=$ ***8***

Find ‘r’.

Solution : We are given that

$N = 10,\ \sigma_K = 9\ \sigma_F = 8$ and $\sum xy = 625$

Now $r = \frac{\sum xy}{N\sigma_X\sigma_Y}$

$= \frac{625}{10\times9\times8}$

$= \frac{625}{720}$

$= 0.868$

$\Rightarrow \quad r = +.868$

Example 6. ***Given***

No. of pairs of observations $= 10$

X ***Series Arithmetic Mean*** $= 75$

Y ***Series Arithmetic Mean*** $= 125$

X ***Series Assumed Mean*** $= 69$

Y ***Series Assumed Mean*** $= 110$

X ***Series Standard Deviation*** $= 13.07$

Y ***Series Standard Deviation*** $= 15.85$

***Summation of products of corresponding deviation of* X *and* Y *series* = 2176 *Find* 'r'.**

Solution : We are given that

$N = 10$, X $= 75$, $A_K = 69$, $\sigma_K = 13.07$

Y $= 125$, $\sigma_F = 15.85$

$A_F = 110$,

and $\sum xy = 2176$

Now $r = \frac{\sum xy - N(\overline{X} - A_X)(\overline{Y} - A_Y)}{N\sigma_X\sigma_Y}$

$= \frac{2176 - 10(75-69)(125-110)}{10 \times 13.07 \times 15.85}$

$= \frac{2176 - 900}{2071.595}$

$= 0.6159 \approx 0.616$

$r = +0.616$

$\Rightarrow$

Example 7. ***A computer while calculating the coefficient of correlation between the variables* X *and* Y *obtained the values as***

$N =$, $\sum X = 50$, $\sum X^2 = 448$

$\sum Y = 106$, $\sum Y^2 = 1896$, $\sum XY = 879$

But later on, it was found that the computer had copied down two pairs of observations as

X	Y
5	15
10	18

While the correct values were

X	Y
6	18
10	19

Find the correct value of correlation coefficient.

Solution : Incorrect value of $\sum$ X $= 50$

∴ Correct value of $\sum X = 50 - 5 - 10 + 6 + 10$

$= 51$ Incorrect

value of $\sum Y = 106$

∴ Correct value of $\sum Y = 106 - 15 - 18 + 18 + 19$

$= 110$

Incorrect value of $\sum X^2 = 448$

∴ Correct value of $\sum X^2 = 448 - 5^2 - (10)^2 + 6^2 + (10)^2$

$= 459$

Incorrect value of $\sum Y^2 = 1896$

∴ Correct value of $\sum Y^2 = 1896 - 15^2 - (18)^2 + (18)^2 + 19^2$

$= 2032$

Incorrect value of $\sum XY = 879$

∴ Correct value of $\sum XY = 879 - (5 \times 15) - (10 \times 18) + (6 \times 18) + (10 \times 19)$

$= 952$

Thus, the corrected value of coefficient of correlation

$$= \frac{N\sum KF - (\sum K)(\sum F)}{\sqrt{N\sum K^2 - (\sum K)^2}\sqrt{N\sum F^2 - (\sum F)^2}}$$

$$= \frac{6\times952 - 51\times110}{\sqrt{6\times459 - (51)^2}\sqrt{6\times2032 - (110)^2}}$$

$$= \frac{5712 - 5610}{\sqrt{2754 - 2601}\sqrt{12{,}192 - 12{,}100}}$$

$$= \frac{102}{\sqrt{153}\sqrt{92}}$$

$$= \frac{102}{12.369\times9.59}$$

$$= \frac{102}{118.618} = 0.8599$$

⇒ $r = +0.8599$

Example 8. Find Correlation between daily wage and food expenditure.

Daily Wage

Food Expenditure	**100-150**	**150-200**	**200-250**	**250-300**	**300-350**
0 –10	**5**	**4**	**5**	**2**	**4**
10 –20	**2**	**7**	**3**	**7**	**1**
20 –30	**-**	**6**	**-**	**4**	**5**
30 –40	**8**	**-**	**4**	**-**	**8**

40 –50	**-**	**7**	**3**	**5**	**10**

Solution:
Assumed Mean of series X = 225
Assumed Mean of series Y = 25
Class interval of series X = 50
Class interval of series Y = 10

Value of dx is calculated by applying the formula = $\frac{m-A}{c}$

Value of dy is calculated by applying the formula = $\frac{m-A}{c}$

CalculationofKarlPearson'scoefficientofcorrelation

X		**100-150**	**150–200**	**200–250**	**250–300**	**300–350**	**f**	**dy**	**fdy**	**fdy^2**	**fdxdy**
Y	**MidPoint**	**125**	**175**	**225**	**275**	**325**					
0–10	**5**	**20** **5**	**8** **4**	**0** **5**	**-4** **2**	**-16** **4**	20	-2	-40	80	8
10–20	**15**	**4** **2**	**7** **7**	**0** **3**	**-7** **7**	**-2** **1**	20	-1	-20	20	2
20–30	**25**	**-** **-**	**0** **6**	**-** **-**	**0** **4**	**0** **5**	15	0	0	0	0
30–40	**35**	**-16** **8**	**-** **-**	**0** **4**	**-** **-**	**16** **8**	20	1	20	20	0
40–50	**45**	**-** **-**	**-14** **7**	**0** **3**	**10** **5**	**40** **10**	25	2	50	100	36
F		15	24	15	18	28	**100 =N**		**10**	**220**	**46**
Dx		-2	-1	0	1	2			**∑fdy**	**∑fdy2**	**∑fdxdy**

Fdx	-30	-24	0	18	56	20	**∑fdx**			
fdx2	60	24	0	18	112	214	**∑fdx2**			
Fdxdy	8	1	0	-1	38	46	**∑fd xdy**			

Coefficient of Correlation, $r =$

$$\frac{N\sum fdx'dy' - (\sum fdx')(\sum fdy')}{\sqrt{N\sum fdx'^2 - (\sum fdx')^2}\sqrt{N\sum fdy'^2 - (\sum fdy')^2}}$$

$$= \frac{100 \times 46 - 20 \times 10}{\sqrt{100 \times 214 - (20)^2}\sqrt{100 \times (220) - (10)^2}}$$

$$= \frac{4600-200}{\sqrt{21400-400}\sqrt{22000-100}}$$

$$= \frac{4400}{\sqrt{21000}\sqrt{21900}}$$

$$= \frac{4400}{21445.28} = .2052$$

9.6.7 ASSUMPTIONS OF KARL PEARSON'S COEFFICIENT OF CORRELATION

1. Linear Relation: It assumes that there exist linear relation between two variable.
2. Causal Relation: It assumes that the relation between variables is not mere chance rather there is cause and effect relation between the variables.
3. Normal Distribution: The series from which data is taken is a normal series.
4. Error of measurement: There is no error in measurement of the data.

9.6.8 BENEFITS OF KARL PEARSON'S COEFFICIENT OF CORRELATION

1. It is most popular method of correlation.
2. It is based on mathematical formula.
3. It give degree of correlation as well as direction of correlation.
4. It is based on all the observations of the data.
5. It is capable of further algebraic treatment.

9.6.9 LIMITATIONS OF KARL PEARSON'S COEFFICIENT OF CORRELATION

1. It is comparatively difficult to calculate.
2. It is time consuming method.

3. It is based on unrealistic assumptions.
4. It is affected by extreme values.
5. It cannot be applied on qualitative data.

9.7 TEST YOUR UNDERSTANDING (A)

1. From the following data of prices of product X and Y draw scatter diagram.

	1	2nd	3	4	5th	6th	7th	8th	9th	10th
Price of X	60	65	65	70	75	75	80	85	80	100
Price of Y	120	125	120	110	105	100	100	90	80	60

st rd th

2. Calculate Karl Pearson's coefficient of correlation

X	21	22	23	24	25	26	27	28	29	30
Y	46	42	38	34	30	26	22	18	14	10

3. Calculate Karl Pearson's coefficient between X and Y

X	42	44	58	55	89	98	66
Y	56	49	53	58	65	76	58

4. Find correlation between marks of subject A Subject B

Subject A	24	26	32	33	35	30
Subject B	15	20	22	24	27	24

5. Find correlation between Height of Mother and Daughter

Height of Mother (Inches)	54	56	56	58	62	64	64	66	70	70
Height of	46	50	52	50	52	54	56	58	60	62

Daughter(Inches)										

6. What is the Karl Pearson's coefficient of correlation if $\sum xy = 40\ , n = 100 , \sum x^2 = 80\ and \sum y^2 = 20$.

7. Calculate the number of items for which r = 0.8 , $\sum xy = 200$. Standard deviation of y = 5 and $\sum x^2 = 100$ where x and y denote the deviations of items from actual means.

8. Following values were obtained during calculation of correlation:

N = 30; ∑X=120 ∑X2 =600 ∑Y=90 ∑Y2=250 ∑XY=335

Later found that two pairs were taken wrong which are as follows:

pairsofobservationsas:	(X,Y):	(8,10)	(12,7)
Whilethecorrectvalueswere:	(X,Y):	(8,12)	(10,8)

Find correct correlation.

9. From the data given below calculate coefficient of correlation.

	X series	Y series
Number of items	8	8
Mean	68	69
Sum of squares of deviation from mean	36	44
Sum, of product of deviations x and y from means	24	24

10. Find the correlation between age and playing habits from the following data :

Age	15	16	17	18	19	20
No of students	20	270	340	360	400	300
Regular players	150	162	170	180	180	120

11. From the table given below find the correlation coefficient between the ages of husbands and wives

Age of Wives Y	Ages of Husbands X					
	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	Total
15 – 25	5	9	3	–	–	17
25 – 35	–	10	25	2	–	37

35 – 45	–	1	12	2	–	15
45 – 55	–	–	4	16	5	25
55 – 65	–	–	–	4	2	6
Total	5	20	44	24	7	100/100

Answers

2. **-1**
3. **.9042**
4. **.92**
5. **.95**
6. **1**
7. **25**
8. **-.4311**
9. **.603**
10. **-.94**
11. **.795**

9.8 SPEARMAN'S RANK CORRELATION

Karl Peason's Coefficient of Correlation is very useful if data is quantitative, but in case of qualitative data it is a failure. Spearman's Rank correlation is a method that can calculate correlation both from quantitative and qualitative data if the data is ranked like in singing contest we rank the participants as one number, two number or three number etc. This method was given by Charles Edward Spearman in 1904. In this method we gives Rank to the data and with help of such ranks, correlation is calculated.

9.8.1 FEATURES OF SPEARMAN'S RANK CORRELATION

1. The coefficient of correlation may be positive or negative.
2. The value of coefficient of correlation always lies between -1 and + 1. -1 refers to 100% negative correlation, plus one refers to 100% positive correlation, and zero refers to no correlation between the items.
3. This method is based ranks of the data.
4. Sum of difference between ranks in this method is always zero i.e. $\sum D = 0$.
5. There is no assumption of normal distribution in this method.
6. In case all the ranks of the two series are same the value of $\sum D^2 = 0$, it shows that there is perfect positive correlation between the data.

9.8.2 SPEARMAN'S RANK CORRELATION WHEN RANKS ARE GIVEN.

1. Calculate the difference between ranks of both the series denoted by $\sum D$.
2. Take square of deviations and calculate the value of D^2.
3. Calculate sum of square of deviations denoted by $\sum D^2$.
4. Apply following formula.

Example 9. *Following are given the ranks of* 8 *pairs. Find 'r'*

Rank X	6	4	8	2	7	5	3	1
Rank *Y*	4	8	7	3	6	5	1	2

Solution :

Rank X	Rank Y	Difference of Ranks D	D^2
6	4	$+2$	4
4	8	-4	16
8	7	-1	1
2	3	-1	1
7	6	$+1$	1
5	5	0	0
3	1	$+2$	4
1	2	-1	1
$N = 8$			$\sum D^2 = 28$

Coefficient of Rank Correlation, $r = 1 - \frac{6\sum D^2}{N(N^2-1)}$

$$= 1 - \frac{6 \times 28}{8(8^2-1)}$$

$$= 1 - \frac{168}{8(64-1)}$$

$$= 1 - \frac{168}{8(63)}$$

$$= 1 - \frac{168}{504} = 1 - 0.33 = 0.67$$

$\Rightarrow$ Rank Correlation Coefficient = 0.67

Example 10. ***In a beauty contest, three judges gave the following ranks to*** 10 ***contestants. Find out which pair of judges agree or disagree the most.***

Judge 1		1	6	3	8	7	10	9	2	4
Judge 2	9	7	10	5	8	4	3	6	1	2
Judge 3	6	4	7	10	5	3	1	9	2	8

Solution :

Ranks by			$D_1 = R_1 - R_2$	D_1^2	$D_2 = R_2 - R_3$	D_2^2	$D_3 = R_1 - R_3$	D_3^2
Judge 1 R_1	Judge 2 R_2	Judge 3 R_3						
5	9	6	-4	16	3	9	-1	1
1	7	4	-6	36	3	9	-3	9
6	10	7	-4	16	3	9	-1	1
3	5	10	-2	4	-5	25	-7	49
8	8	5	0	0	3	9	3	9
7	4	3	$+3$	9	1	1	4	16
10	3	1	$+7$	49	2	4	9	81
9	6	9	$+3$	9	-3	9	0	0
2	1	2	$+1$	1	-1	1	0	0
4	2	8	$+2$	4	-6	36	-4	16
				$\sum D_1^2 = 144$		$\sum D_2^2 = 112$		$\sum D_3^2 = 182$

Now $r_{12} = 1 - \frac{6\sum D1^2}{N(N^2-1)}$

$= 1 - \frac{6\times144}{10(10^2-1)}$

$= 1 - \frac{864}{10(100-1)}$

$= 1 - \frac{864}{10(99)}$

$= 1 - \frac{864}{990}$

$= 1 - 0.873 = 0.127$

∴ $r_{12} = +0.127 \Rightarrow$ Low degree $+ve$ correlation

$r_{23} = 1 - \frac{6\sum D2^2}{N(N^2-1)}$

$= 1 - \frac{6\times112}{10(10^2-1)}$

$= 1 - \frac{672}{10(100-1)}$

$= 1 - \frac{672}{10(99)}$

$= 1 - \frac{672}{990}$

$= 1 - 0.679 = 0.321$

∴ $r_{23} = +0.321 \Rightarrow$ Moderate degree $+ve$ correlation

Similarly $r_{31} = 1 - \frac{6\sum D3^2}{N(N^2-1)}$

$= 1 - \frac{6\times182}{10(10^2-1)}$

$= 1 - \frac{1092}{10(100-1)}$

$= 1 - \frac{1092}{10(99)}$

$= 1 - \frac{1092}{990}$

$= 1 - 1.103 = -0.103$

∴ $r_{31} = -0.103 \Rightarrow$ Low degree $-ve$ correlation

⇒ Since r_{23} is highest, so $2nd$ and $3rd$ judges agree the most.

Also, r_{31} being lowest, $3rd$ and $1st$ judges disagree the most.

9.8.3 SPEARMAN'S RANK CORRELATION WHEN RANKS ARE NOT GIVEN.

1. Assign the ranks in descending order to series X by giving first rank to highest value and second rank to value lower than higher value and so on.
2. Similarly assign the ranks to series Y.
3. Calculate the difference between ranks of both the series denoted by $\sum D$.
4. Take square of deviations and calculate the value of D^2.
5. Calculate sum of square of deviations denoted by $\sum D^2$.
6. Apply following formula.

Example 11. ***Following are the marks obtained by*** 8 ***students in Maths and Statistics. Find the Rank Correlation Coefficient.***

Marks in Maths	60	70	53	59	68	72	50	54
Marks in Statistics	44	74	54	64	84	79	53	66

Solution :

X	Ranks R_1	Y	Ranks R_2	Difference of Ranks $= R_1 - R_2$	D^2
60	4	44	8	-4	16
70	2	74	3	-1	1
53	7	54	6	$+1$	1
59	5	64	5	0	0
68	3	84	1	$+2$	4
72	1	79	2	-1	1
50	8	53	7	$+1$	1
54	6	66	4	$+2$	4
					$\sum D^2 = 28$

Here $N = 8$

$\Rightarrow$ Rank Coefficient of Correlation, $r = 1 - \frac{6\sum D^2}{N(N^2-1)}$

$$= 1 - \frac{6\times 28}{8(8^2-1)}$$

$$= 1 - \frac{168}{8(64-1)}$$

$$= 1 - \frac{168}{8(63)}$$

$$= 1 - \frac{168}{504}$$

$$= 1 - 0.33 = 0.67$$

$\Rightarrow$ Rank Correlation Coefficient = 0.67

9.8.4 SPEARMAN'S RANK CORRELATION WHEN THERE IS REPETITION IN RANKS.

1. Assign the ranks in descending order to series X by giving first rank to highest value and second rank to value lower than higher value and so on. If two items have same value, assign the average rank to both the item. For example two equal values have ranked at 5^{th} place than rank to be given is 5.5 to both i.e. mean of 5^{th} and 6^{th} rank. **($\frac{5+6}{2}$).**
2. Similarly assign the ranks to series Y.
3. Calculate the difference between ranks of both the series denoted by $\sum D$.
4. Take square of deviations and calculate the value of D^2.
5. Calculate sum of square of deviations denoted by $\sum D^2$.
6. Apply following formula.

$$r = 1 - \frac{6\{\sum D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2)\}}{N(N^2 - 1)}$$

Where m = no. of times a particular item is repeated.

Example 12. ***Find the Spearsman's Correlation Coefficient for the data given below***

X	110	104	107	82	93	93	115	95	93	113
Y	80	78	90	75	81	70	87	78	73	85

Solution : Here, in X series the value 93 occurs thrice ($m_1 = 3$), i. $e.$ at 7^{th}, 8^{th} and 9^{th} rank. So all the three values are given the same average rank, i.$e.$ $\frac{7+8+9}{3} = 8^{th}$rank.

Similarly, in Y series the value 78 occurs twice ($m_2 = 2$), i. $e.$ at 6^{th} and 7^{th} rank. So both the values are given the same average rank, i.$e.$ $\frac{6+7}{2} = 6.5^{th}$rank.

X	Ranking of X R_1	Y	Ranking of Y R_2	Difference of Ranks $D = R_1 - R_2$	D_2
110	3	80	5	-2	4

104	5	78	6.5	-1.5	2.25
107	4	90	1	$+3$	9
82	10	75	8	$+2$	4
93	8	81	4	$+4$	16
93	8	70	10	-2	4
115	1	87	2	-2	1
95	6	78	6.5	-0.5	0.25
93	8	73	9	-1	1
113	2	85	3	-1	1
					$\sum D^2 = 42.5$

Here $N = 10$

Spearman's Rank Correlation Coefficient, $r = 1 - \frac{6\{\sum D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2)\}}{N(N^2-1)}$ i.

e. $r = 1 - \frac{6\{42.50 + \frac{1}{12}(3^3-3) + \frac{1}{12}(2^3-2)\}}{10(10^2-1)}$

$= 1 - \frac{6\{42.50 + \frac{24}{12} + \frac{6}{12}\}}{10(100-1)}$

$= 1 - \frac{6\{42.50 + 2 + \frac{1}{2}\}}{10 \times 99}$

$= 1 - \frac{6\{42.5 + 2.5\}}{990}$

$= 1 - \frac{6 \times 45}{990}$

$= 1 - 0.2727 = 0.7273$

⇒ Rank Correlation Coefficient = 0.7273

Example 13. ***The rank correlation coefficient between the marks obtained by ten students in Mathematics and Statistics was found to be*** 0.5. ***But later on, it was found that the difference in ranks in the two subjects obtained by one students was wrongly taken as*** 6 ***instead of*** 9.

Find the correct rank correlation.

Solution : Given $N = 10$, Incorrect $r = 0.5$

We know that

Rank Correlation Coefficient, $r = 1 - \frac{6\sum D^2}{N(N^2-1)}$

$$= 1 - \frac{6\sum D^2}{10(10^2-1)} = 1 - \frac{6\sum D^2}{10\times 99}$$

⇒ 0.5

⇒ Incorrect $\sum D^2 = \frac{990}{6} \times 0.5 = 82.5$

∴ The corrected value of $\sum D^2 = 82.5 - 6^2 + 9^2$

$$= 82.5 - 36 + 81 = 127.5$$

∴ Correct Rank Correlation Coefficient , $r = 1 - \frac{6\times 127.5}{10(10^2-1)}$

$$= 1 - \frac{765}{10(100-99)}$$

$$= 1 - \frac{765}{10\times 99}$$

$$= 1 - \frac{765}{990}$$

$$= 1 - 0.7727$$

$$= 0.2273$$

9.8.5 MERITS OF SPEARMAN'S RANK CORRELATION

1. This is easy to understand.
2. It can calculate correlation from qualitative data also.
3. It does not put condition of normal series of data.
4. It can deal with quantitative data also.
5. It is not affected by presence of extreme values.

9.8.6 LIMITATIONS OF SPEARMAN'S RANK CORRELATION

1. It cannot deal with grouped data.
2. If large data is there, it is difficult to apply this method.
3. It cannot be applied further algebraic treatement.
4. Combined correlation cannot be calculated.
5. It gives only approximate correlation, it is not based on actual values.

9.9 TEST YOUR UNDERSTANDING (A)

1. Find Rank correlation on base of following data.

X	78	36	98	25	75	82	90	62	65	39

Y	84	51	91	60	68	62	86	58	53	47

In Dance competition following ranks were given by 3 judges to participants. Determine which two judges have same preference for music:

1stJudge	1	6	5	10	3	2	4	9	7	8
2ndJudge	3	5	8	4	7	10	2	1	6	9
3rdJudge	6	4	9	8	1	2	3	10	5	7

3. Find Rank correlation on base of following data.

X	25	30	38	22	50	70	30	90
Y	50	40	60	40	30	20	40	70

4. Find Rank correlation on base of following data.

X	63	67	64	68	62	66	68	67	69	71
Y	66	68	65	69	66	65	68	69	71	70

Answers

1. **.82**
2. **I and II -.2121, II and III -.297, I and III .6364, so judge I and III**
3. **0**
4. **.81**

9.10 LET US SUM UP

- **Correlation shows the relation between two or more variables.**
- **Value of the coefficient of correlation always lies between -1 and +1.**
- **Correlation may be positive or negative.** • **Correlation may be linear or non linear.**
- **Karl Person's coefficient of correlation is the most popular method of correlation.**
- **It can deal only with quantitative data.**
- **Spearman's Rank correlation calculated correlation on the basis of ranks given to data.**
- **It can deal with qualitative data also.**

9.11 KEY TERMS

- **CORRELATION:**Correlation is a statistical technique which studies the relation between two or more variables. It studies that how to variables are related to each other.
- **POSITIVE CORRELATION**: It is a situation in which two variables move in the same direction. In this case if the value of one variable increase the value of other variable also increase. Similarly, if the value of one variable decrease, the value of other variable also decrease.
- **NEGATIVE OR INVERSE CORRELATION:**When two variables move in opposite direction from each other, it is known as negative or inverse correlation. In other words, we can say that when the value of one variable increase value of other variable decrease, it is called negative correlation.
- **LINEAR CORRELATION:** When the change in value of one variable results into constant ratio of change in the value of other variable, it is called linear correlation. In such case if we draw the values of two variables on the graph paper, all the points on the graph paper will fall on a straight line.
- **NON - LINEAR CORRELATION:** When the change in value of one variable does not result into constant ratio of change in the value of other variable, it is called non linear correlation. In such case, if we draw the value of two variables on the graph paper all the points will not fall in the straight line on the graph.
- **SIMPLE CORRELATION:** When we study relation between two variables only, it is known as simple correlation. For example, relation between income and expenditure, Price and Demand, are situations of simple correlation.
- **MULTIPLE CORRELATION:** Multiple correlation is a situation in which more than two variables are involved. Here relation between more than two variables are studied together, for example if we are studying the relation between income of the consumer, price if the product and demand of the product, it is a situation of multiple correlation.

9.12 REVIEW QUESTIONS

10. What is Correlatlion. What are uses of measuring correlation.
11. Give different types of correlation.
12. Give Karl Persons method of calculating correlation.
13. Give Karl Pearson's coefficient of correlation in case of actual and assumed mean.
14. What are merits and limitations of Karl Pearson's method.
15. What is Spearman's Rank correlation. How it is determined.
16. In case of repeated ranks how would you determine Spearman's Rank correlation.
17. What are the merits and limitations of Spearman's Rank correlation

9.13 FURTHER READINGS

1. J. K. Sharma, *Business Statistics*, Pearson Education.
2. S.C. Gupta, *Fundamentals of Statistics*, Himalaya Publishing House.
3. S.P. Gupta and Archana Gupta, *Elementary Statistics*, Sultan Chand and Sons, New Delhi.
4. Richard Levin and David S. Rubin, *Statistics for Management*, Prentice Hall of India, New Delhi.
5. M.R. Spiegel, *Theory and Problems of Statistics*, Schaum's Outlines Series, McGraw HillPublishing Co.

B. COM (DIGITAL)

SEMESTER II

COURSE: BUSINESS MATHEMATICS AND STATISTICS

UNIT 10 – REGRESSION

STRUCTURE

10.0 OBJECTIVES

After studying the Unit, students will be able to

- Describe what is regression.
- Distinguish between different types of Regression.
- Understand benefits of Regression.
- Find Regression using various methods.
- Show how correlation and regression are related.
- Understand properties of regression coefficients.

10.1 INTRODUCTION

Statistics has many applications in our life whether its business life or our routine life. There are many techniques in statistics that can help us in prediction. Regression is one such technique. In the literary meaning the term 'Regression' 'going back', or 'stepping down'. So, the regression analysis is a tool in statistics that can help in prediction of one variable when the value of other variable is known if there exists any close relation between two or more variables, though such relation may be positive or negative. The technique of Regression can be widely used used as a powerful tool in almost all the fields whether science, Social science, Business etc. However, particularly, in the fields of business and management this technique is very useful for studying the relationship between different variables scuh as, Price and Demand, Price and Supply, Production and Consumption, Income and Consumption, Income and Savings, etc.

When we find regression between two or more varibales, we try to understand the behavior of one variable with help movement of the other variable in a particular direction. For example, if the correlation coefficient between value of sales and amount spent on advertisement say +0.9, it means that if advertisement expenditure is increased, Sale is also likely to increase, as there is very high positive relation between the two variables. However, correlation only tells relation between two variables, but it does not tells the extent by which change in one variable will affect the change in other variable. For this purpose, we have to calculate the co-efficient of Regression. Regression Coefficient is a statistical measure that tries to find out the value of one variable known as dependent variable when the value of another variable known as independent variable is known. Thus, in case of two variables, like Advertisement expenditure and amount of Sales, we can estimate the likely amount of Sales if the value of Advertisement expenditure is given. Similarly, we can predict the value of Advertisement expenditure required, to achieve a particular amount of Sales. This can be done using the two regression coefficients

10.2 HISTORY OF REGRESSION ANALYSIS

The technique of Regression analysis was developed by the British Biometrician Sir Francis Galton in 1877 while he was studying the relationship between the heights of fathers and the heights of their sons. The term' regression' was first time used by him in his paper 'Regression towards Mediocrity in Hereditary Stature" in which he said that:

(i) That tall fathers will most probably have tall sons, and short fathers will most probably have short sons; and the average height of the sons of tall fathers' will mostly be less than the average height of their fathers ;

(ii) He also said that the average height of the sons of short fathers' is most likely to be more than the average height of their fathers ; and

(iii) That the deviations of the mean height of the sons is most likely to be less than the deviations of the mean height of the, or that when the fathers' height move above or below the mean, the sons' height tend to go back (regress) towards the mean.

Professor Galton in his study analyzed the relationship between the two variables that is the heights of the fathers and the heights of the sons using the graphical technique and named the such line describing the relationship between height of the father and height of the son as the 'Line of Regression'.

10.3 MEANING OF REGRESSION ANALYSIS

Many experts have defined the term Regression in their own way. Some of these definitions are given below:

1. According to Sir Francis Galton, the term regression analysis is defined as "the law of regression that tells heavily against the full hereditary transmission of any gift, the more bountifully the parent is gifted by nature, the more rare will be his good fortune if he begets a son who is richly endowed as himself, and still more so if he has a son who is endowed yet more largely."

2. In the words of Ya Lun Chou, "Regression analysis attempts to establish the nature of the relationship between variables that is to study the functional relationship between the variables and thereby provide mechanism for prediction or forecasting".

10.4 BENEFITS / USES OF REGRESSION ANALYSIS

Benefits of Regression analysis are outlined as under :

1. **Forecasting or Prediction** – Regression provides a relationship between two or more variables that are related to each other. So, with the help of this technique we can easily forecast the values of one variable which is unknown from the values of another variable which is known.
2. **Cause and Effect Relationship –** This analysis helps in finding the cause and effect relationship between two or more variables. It is a powerful tool for measuring the cause and effect relationship among the economic variables. In the field of economics it is very beneficial in the estimation of Demand, Production, Supply etc.

3. **Measuring Error in Estimation –** Regression helps in measuring of errors in estimates made through the regression lines. In case the point of Regression line are less scattered around the relevant regression line, it means there are less chances of error but if these are more scattered around line of regression, it means there are more chances of error.
4. **Finding Correlation Coefficient between two variables** – Regression provides a measure of coefficient of correlation between the two variables. We can calculate correlation by taking the square root of the product of the two regression coefficients.
5. **Usefulness in Business and Commerce** – Regression is very powerful tool of statistical analysis in the field of business and commerce as it can help businessman in prediction of various values such as demand, production etc.
6. **Useful in day to day life -** This technique is very useful in our daily life as it can predict the various factors such as birth rate, death rate, etc.
7. **Testing Hypothesis –** The technique of regression can be used in testing the validity of economic theory or testing of any hypothesis.

10.5 LIMITATIONS OF REGRESSION ANALYSIS

Though Regression is a wonderful statistical tool, still it suffers from some limitations. Following are limitations of Regression analysis:

1. Regression analysis assumes that there exists cause and effect relationship between the variables and such relation is not changeable. This assumption may not always hold good and thus could give misleading results.
2. Regression analysis is based on some limited data available. However, as the values are based on limited data may it may give misleading results.
3. Regression analysis involves very lengthy and complicated steps of calculations and analysis. A layman may not be in a position to use this technique.
4. Regression analysis can be used in case of quantitative data only. It cannot be used where data is of qualitative nature such as hardwork, beauty etc.

10.6 DIFFERENT TYPES OF REGRESSION ANALYSIS

1. SIMPLE AND MULTIPLE REGRESSION

- **SIMPLE REGRESSION**: When there are only two variables under study it is known as a simple regression. For example we are studying the relation between Sales and

Advertising expenditure. If we consider sales as Variable X and advertising as variable Y, then the X = a+bY is known as the regression quation of X on Y where X is the dependent variables and Y and the independent variable. In other words we can find the value of variable X (Sales) if the value of Variable Y (Advertising) is given.

- **MULTIPLE REGRESSION:**The study of more than two variables at a time is known as multiple regression. Under this, only one variable is taken as a dependent variable and all the other variables are taken as independent variables. For example If we consider sales as Variable X, advertising as variable Y and Income as Variable Z, then using the functional relation X = *f*(Y, Z), we can find the value of variable X (Sales) if the value of Variable Y (Advertising) and the value of variable Z (Income) is given.

2. TOTAL AND PARTIAL REGRESSION

a. TOTAL REGRESSION: Total regression analysis is one in which we study the effect of all the variables simultaneously. For example, when we want to study the effect of advertising expenditure of business represented by variable Y, income of the consumer represented by variable Z, on the amount of sales represented by variable X, we can study impact of advertising and income simultaneously on sales. This is a case of total regression analysis. In such cases, the regression equation is represented as follows:

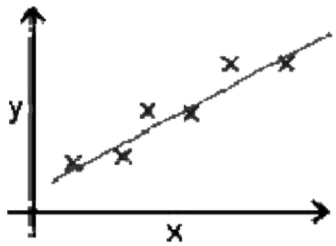
X = *f*(Y, Z),

a. PARTIAL REGRESSION: In the case of Partial Regression one or two variables are taken into consideration and the others are excluded. For example, when we want to study the effect of advertising expenditure of business represented by variable Y, income of the consumer represented by variable Z, on the amount of sales represented by variable X, we will not study impact of both income and advertising simultaneously, rather we will first study effect of income on sales keeping advertising constant and then effect of advertising on sales keeping income constant. Partial regression can be written as **X=f(Y not Z).**

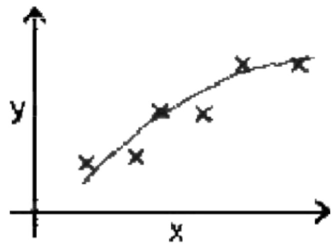
3. LINEAR AND NON-LINEAR REGRESSION

a. LINEAR REGRESSION: When the functional relationship between X and Y is expressed as the first degree equations, it is known as linear regression. In other words, when the points plotted on a scatter diagram concentrate around a straight line it is the case of linear regression.

b. NON-LINEAR REGRESSION: On the other hand if the line of regression (in scatter diagram) is not a straight line, the regression is termed as curved or non-linear regression. The regression equations of non-linear regression are represented by equations of higher degree. The following diagrams show the linear and non-linear regressions:



Linear regression



Nonlinear regression

10.7 RELATIONSHIP BETWEEN CORRELATION AND REGRESSION

1. Correlation is a quantitative tool that measure of the degree of relationship that is present between two variables. It shows the degree and direction of the relation between tow variables. Regression helps us to find the value of a dependent variable when the value of independent variable is given.
2. Correlation between two variables is the same. For example we calculate the correlation between sales and advertising or advertising and sales, the value of correlation will remain same. But this is not true for Regression. Regression equation of Advertising on sales will be different from regression equation of Sales on advertising.
3. If there is positive correlation, the distance between the two lines will be less. That means the two regression lines will be closer to each other- Similarly, if there is low correlation, the lines will be farther to each other. A positive correlation implies that the lines will be upward sloping whereas a negative correlation implies that the regression lines will be downward sloping.
4. Correlation between two variables can be calculated by taking the square root of the product of the two regression coefficients.

Following are some of the difference between Correlation and Regression:

No.	Correlation	Regression
1.	Correlation measures the degree and direction of relationship between two variables.	Regression measures the change in value of a dependent variable given the change in value of an independent variable.
2.	Correlation does not depicts a cause and effect' relationship.	Regression depicts the causal relationship between two variables.
3.	Correlation is a relative measure of linear relationship that exists between two variables.	Regression is an absolute measure which measures the change in value of a variable.
4.	Correlation between two variables is the same. . In other words, Correlation between two variables is the same. rxy =ryx.	Regression is not symmetrical in formation. So, the regression coefficients of X on Y and of Y on X are different.
5.	Correlation is independent of Change in origin or scale.	Regression is independent of Change in origin but not of scale.

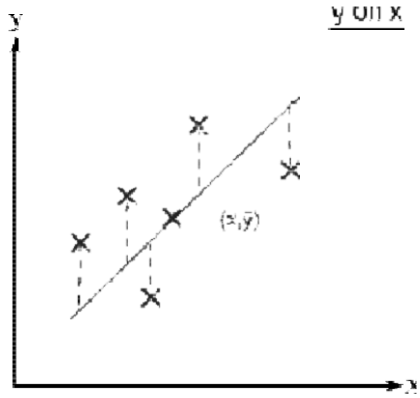
6.	Correlation is not capable of any further mathematical treatment.	Regression can be further treated mathematically.
7.	Coefficient of correlation always lies between -1 and +1.	Regression coefficient can have any value.

10.8 REGRESSION LINES

The lines that are used in Regression for the purpose of estimation are called as regression line. In other words, the lines that are used to study the dependence of one variable on the other variable are called as regression line. If we have two variables X and Y then there .

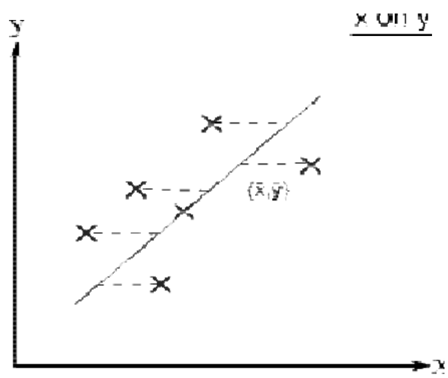
a. REGRESSION LINE OF Y ON X: Regression Line Y on X measures the dependence of Y on X and we can estimate the value of Y for the given values of X. In this line Y is dependent

variable and X is independent variable.

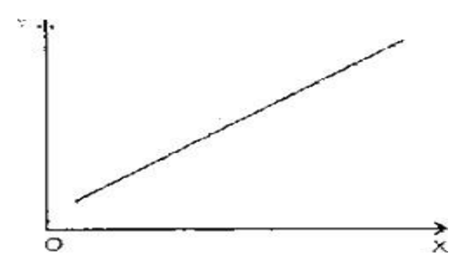


b. Regression Line of X on Y: Regression Line X on Y measures the dependence of X on Y and we can estimate the value of X for the given values of Y. In this line X is dependent variable and

Y is independent variable.



The direction of two regression equation depends upon the degree of correlation between two

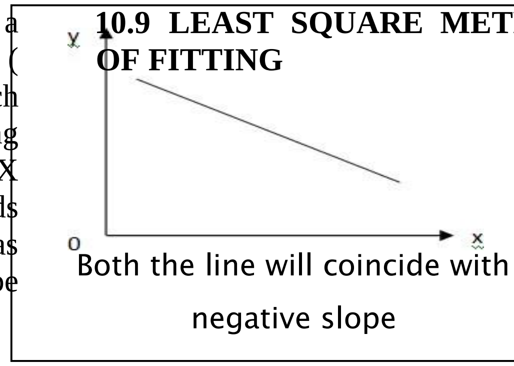


Both the line will coincide with positive slope variables. Following can be the cases of correlation between two variables:

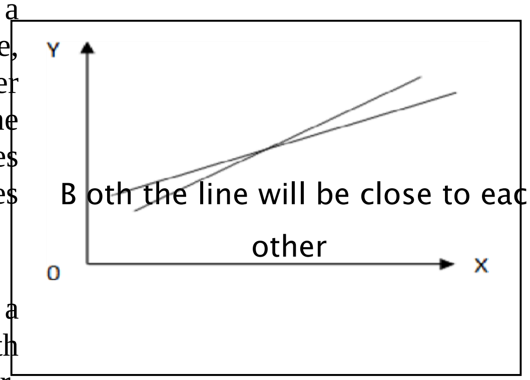
1. Perfect positive correlation: If there is a perfect positive correlation between two variable (

i.e. r = +1), both the lines will coincide with each other and will be having positive slope. Both the lines X on Y and Y on X will be same in this case. In other words in that case only one regression line can be drawn as shown in the diagram. The slope of the line will be upward.

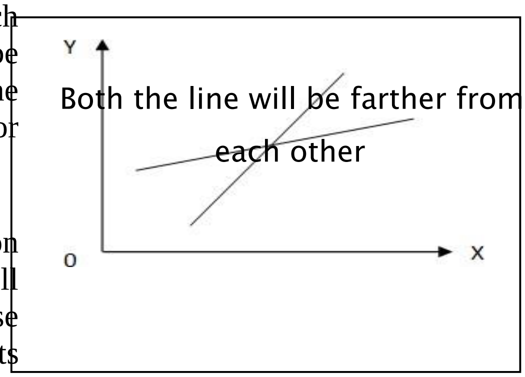
2. Perfect negative correlation: If there is a perfect negative correlation between two variable (i.e. r = -1), both the lines will coincide with each other and will in such case these lines will be having negative slope. Both the lines X on Y and Y on X will be same but downward sloping. In other words in that case only one regression line can be drawn as shown in the diagram. The slope of the line will be upward.



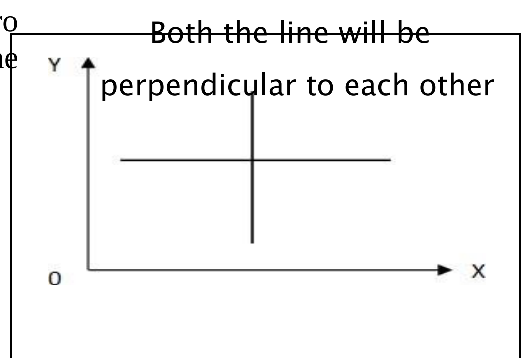
3. High degree of correlation: If there is a High degree of correlation between two variable, both the lines will be near to each other. In other words these lines will be closer to each other but the lines will not coincide with each other. Both the lines will be separate. Further the direction of lines depends upon the positive or negative correlation.



4. Low degree of correlation: If there is a low degree of correlation between two variable, both the lines will be have more distance from each other. In other words these lines will be farther to each other, that is the gap between the two lines will be more. Both the lines will be separate. Further the direction of lines depends upon the positive or negative correlation.



5. No correlation: If there is a no correlation between two variable (i.e. r = 0), both the lines will be perpendicular to each other. In other words these lines will cut each other at 90^{0}. This diagram depicts the perpendicular relation between the two regression lines when there is absolutely zero correlation between the two variables under the study.



10.9 LEAST SQUARE METHOD OF FITTING

REGRESSION LINES

Under this method the lines of best fit is drawn as the lines of regression. These lines of regression are known as the lines of the best fit because, with help of these lines we can make the estimate of the values of one variable depending on the value of other variable. According to the Least Square method, regression line should be plotted in such a way that sum of square of the difference between actual value and estimated value of the dependent variable should be least or minimum possible. Under this method we draw two regression lines that are

a. **REGRESSION LINE Y ON X**– it measures the value of Y when value of X is given. In other words it assumes that X is an Independent variable whereas the other variable Y is dependent variable. Mathematically this line is represented by

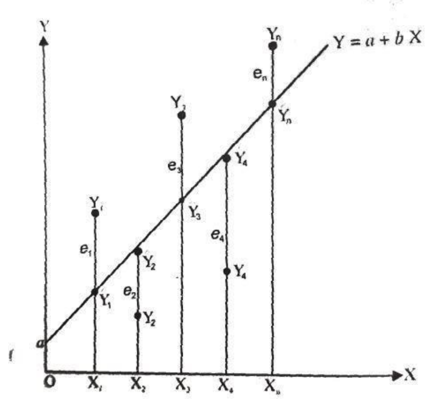
$$Y = a + bX$$

Where Y – Dependent Variable X
– Independent Variable a
& b – Constants

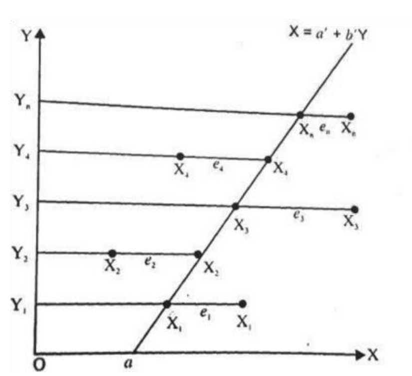
b. **REGRESSION LINE X ON Y** – it measures the value of X when value of Y is given. In other words it assumes that Y is an Independent variable whereas the other variable X is dependent variable. Mathematically this line is represented by

$$X = a + bY$$

Where X – Dependent Variable Y
– Independent Variable a
& b – Constants



Equation Y on X



Equation X on Y

In the above two regression lines, there are two constants represented by “a” and “b”. The constant “b” is also known as regression coefficient, which are denoted as “byx” and “bxy”, Where “byx” represent regression coefficient of equation Y on X and “bxy” represent regression coefficient of equation X on Y . When the value of these two variable “a” and “b” is determined we can find out the regression line.

10.10 DIRECT METHODS TO ESTIMATE REGRESSION EQUATION

The regression equations can be obtained by 'Normal Equation Method" as follows:

1. **REGRESSION EQUATION OF Y ON X:** The regression equation Y on X is in the format of Y = a + bX, where Y is a Dependent Variable and X is an Independent Variable. To estimate this regression equation, following normal equations are used: $\Sigma Y = na + b_{yx}\Sigma X$ $\Sigma XY = a\Sigma X + b_{yx}\Sigma X^2$

   With the help of these two equations the values of ‘a' and 'b' are obtained and by putting the values of ‘a' and 'b' in the equation Y = a + bX we can predict or estimate value of Y for any value of X.
2. **REGRESSION EQUATION OF X ON Y:** The regression equation X on Y is in the format of X = a + bY, where X is a Dependent Variable and Y is an Independent Variable. To estimate this regression equation, following normal equations are used: $\Sigma X = na + b_{xy}\Sigma Y$

   $\Sigma XY = a\Sigma Y + b_{xy}\Sigma Y^2$

   With the help of these two equations the values of ‘a' and 'b' are obtained and by putting the values of ‘a' and 'b' in the equation X = a + bY we can predict or estimate value of Y for any value of X.

Example 1. Find out the two regression lines for the data given below using the method of least square.

Variable X :	5	10	15	20	25
Variable Y :	20	40	30	60	50

Determination of the regression lines by the method of least square. Also find out

a. Value of Y when value of X is 40
b. Value of X when value of Y is 80.

Solution:

X	Y	X2	Y2	XY
5	20	25	400	100
10	40	100	1600	400
15	30	225	900	450
20	60	400	3600	1200

25	50	625	2500	1250
XX = 75	XY = 200	XX2=1375	XY2=9,000	XXY =3400

(i) REGRESSION LINE OF Y ON X

This is given by Y = *a + bX* where *a* and *b* are the two constants which are found by solving simultaneously the two normal equations as follows :

$\Sigma Y = na + b_{yx}\Sigma X$ ΣXY
$= a \Sigma X + b_{yx}\Sigma X^2$

Substituting the given values in the above equations we get,
200 = 5a + 75b .... (i)
3400 =75a + 1375b .... (ii)
Multiplying the eqn. (i) by 15 we get
3000 = 75a + 1125b .... (iii)

Subtracting the equation (iii) from equation (ii) we get,
3400 =75a + 1375b
-3000 = -75a - 1125b
400 = 250b

or b = 1.6

Putting the above value of b in the eqn. (i) we get,
200 = 5a+ 75(1.6) or 5a
=200- 120 or
a = 16
Thus, a = 16, and b = 1.6

Putting these values in the equation Y = *a* + bX we get
Y = 16+ 1.6X

So when X is 40, the value of Y will be
Y = 16+ 1.6(40) = 80

(ii) REGRESSION LINE OF X ON Y

This is given by X = *a + bY*
where *a* and *b* are the two constants which are found by solving simultaneously the two normal equations as follows :

$\Sigma X = na + b_{xy}\Sigma Y$

$\Sigma XY = a\, \Sigma Y + b_{xy}\Sigma Y^2$

Substituting the given values in the above equations we get,

75 = 5a+200b .... (i)

3400 = 200a + 9000b .... (ii)

Multiplying the eqn. (i) by 40 we get

3000 = 200a + 8000b .... (iii)

Subtracting the equation (iii) from equation (ii) we get,

3400 = 200a + 9000b

-3000 = -200a + -8000b

400 = 1000b

or b = .4

Putting the above value of b in the eqn. (i) we get,

75 = 5a + 200(.4) or

5a = -5 or

a = -1

Thus, a = -1, and b = .4

Putting these values in the equation X = *a + bY* we get

X = -1+ .4Y

So when Y is 80, the value of X will be

X = -1+ .4(80) = 31

10.11 OTHER METHODS OF ESTIMATING REGRESSION EQUATION

This method discussed above is known as direct method. This is one of the popular method of finding the regression equation. But sometime this method of finding regression equation becomes cumbersome and lengthy specially when the values of X and Y are very large. In this case we can simplify the calculation by take the deviations of X and Y thatn dealing with actual values of X and Y. In such case

Regression equation Y on X

Y = a + bX

will be converted to $(Y - \bar{Y}) = byx\,(X - \bar{X})$

Similarly Regression equation X on Y:

$X = a + bY$

will be converted into $(X - \bar{X}) = bxy\,(Y - \bar{Y})$

Now when we are using these regression equations, the calculations will become very simple as now we have to calculate value of only one constant that is value of "b" which is our regression coefficient. As there are two regression equations, so we need to calculate two regression coefficients that is Regression Coefficient X on Y, which is symbolically denoted as "bxy" and similarly Regression Coefficient Y on X, which is denoted as "byx". However, these coefficients can also be calculated using different methods. As we take deviations under this method, we can take deviations using actual mean, assumed mean or we can calculate it by not taking the deviations. Following formulas are used in such cases:

Method	**Regression Coefficient X on Y**	**Regression Coefficient Y on X**
When deviations are taken from actual mean	$bxy = \frac{\sum xy}{\sum y^2}$	$byx = \frac{\sum xy}{\sum x^2}$
When deviations are taken from assumed mean	$bxy = \frac{N\sum dxdy - \sum dx \sum dy}{N\sum dy^2 - (\sum dy)^2}$	$byx = \frac{N\sum dxdy - \sum dx \sum dy}{N\sum dx^2 - (\sum dx)^2}$
Direct Method: Using sum of X and Y	$bxy = \frac{N\sum XY - \sum X \sum Y}{N\sum Y^2 - (\sum Y)^2}$	$byx = \frac{N\sum XY - \sum X \sum Y}{N\sum X^2 - (\sum X)^2}$
Using the correlation coefficient (r) and standard deviation (σ)	$bxy = r \cdot \frac{\sigma x}{\sigma y}$	$byx = r \cdot \frac{\sigma y}{\sigma x}$

Example 2. Fromthe information give below obtaintworegressionlines X on Y and Y on X using

1. Actual Mean Method.
2. Assumed Mean Method
3. Direct Method (Without taking Mean)

Number of Hrs Machine Operated	7	8	6	9	11	9	10	12
Production(Units in 000):	4	5	2	6	9	5	7	10

Solution:

1. **Actual Mean Method**

CalculationofRegressionEquation

X	**Y**	**x =X - X̄**	**x2**	**y =Y - Ȳ**	**y2**	**xy**
7	4	-2	4	-2	4	4
8	5	-1	1	-1	1	1
6	2	-3	9	-4	16	12
9	6	0	0	0	0	0

11	9	2	4	3	9	6
9	5	0	0	-1	1	0
10	7	1	1	1	1	1
12	10	3	9	4	16	12
$\sum X = 72$	**$\sum Y = 48$**		**$\sum x^2 = 28$**		**$\sum y^2 = 48$**	**$\sum xy = 36$**

$$\bar{X} = \frac{\Sigma X}{N} = \frac{72}{8} = 9$$

$$\bar{Y} = \frac{\Sigma Y}{N} = \frac{48}{8} = 6$$

Regression equation of X on Y:

$(X - \bar{X}) = b_{xy} (Y - \bar{Y})$

Where bxy = $\frac{\Sigma xy}{y^2}$

$= \frac{36}{48}$

$= .75$

So $(X - 9) = .75 (Y - 6)$

$X - 9 = .75Y - 4.5$

X = 4.5 + .75Y

e gression equation o f Y on X:

$(Y - \bar{Y}) = b_{xy} (X - \bar{X})$

Where byx = $\frac{\Sigma xy}{x^2}$

$= \frac{36}{28}$

$= 1.286$

So $(Y - 6) = 1.286 (X - 9)$

$Y - 6 = 1.286X - 11.57$

Y = – 5.57+ 1.286X

2. Assumed Mean Method

CalculationofRegressionEquation

X	**Y**	**dx =X – A (A = 8)**	**dx^2**	**dy =Y – A (A = 5)**	**dy^2**	**dxdy**
7	4	-1	1	-1	1	1
8	5	0	0	0	0	0

6	2	-2	4	-3	9	6
9	6	1	1	1	1	1
11	9	3	9	4	16	12
9	5	1	1	0	0	0
10	7	2	4	2	4	4
12	10	4	16	5	25	20
$\sum X = 72$	**$\sum Y = 48$**	**$\sum dx = 8$**	**$\sum dx^2 = 36$**	**$\sum dy = 8$**	**$\sum dy^2 = 56$**	**$\sum xy = 44$**

$\bar{X} = \frac{\sum X}{N} = \frac{72}{8} = 9$

$\bar{Y} = \frac{\sum Y}{N} = \frac{48}{8} = 6$

Regression equation of X on Y:

$(X - \bar{X}) = b_{xy} (Y - \bar{Y})$

Where bxy = $\frac{N\sum dxdy - \sum dx \sum dy}{N\sum dy^2 - (\sum dy)^2}$

$= \frac{8\,(44) - (8)\,(8)}{8\,(56) - (8)^2}$

$= \frac{352 - 64}{448 - 64}$

$= \frac{288}{384}$

$= .75$

So $(X - 9) = .75\,(Y - 6)$ $X - 9 = .75Y - 4.5$

$X = 4.5 + .75Y$

Regression equation of Y on X:

$(Y - \bar{Y}) = b_{xy} (X - \bar{X})$

Where byx = $\frac{N\sum dxdy - \sum dx \sum dy}{N\sum dx^2 - (\sum dx)^2}$

$= \frac{8\,(44) - (8)\,(8)}{8\,(36) - (8)^2}$

$= \frac{288}{224}$

$= 1.286$

So $(Y - 6) = 1.286(X - 9)$

$Y - 6 = 1.286X - 11.57$

Y = – 5.57+ 1.286X

3. DIRECT METHOD (WITHOUT TAKING MEAN)

CalculationofRegressionEquation

X	Y	X2	Y2	XY
7	4	49	16	28
8	5	64	25	40
6	2	36	4	12
9	6	81	36	54
11	9	121	81	99
9	5	81	25	45
10	7	100	49	70
12	10	144	100	120
$\sum X = 72$	**$\sum Y = 48$**	**$\sum X^2 = 676$**	**$\sum Y^2 = 336$**	**$\sum XY = 468$**

$$\bar{X} = \frac{\sum X}{N} = \frac{72}{8} = 9$$

$$\bar{Y} = \frac{\sum Y}{N} = \frac{48}{8} = 6$$

REGRESSION EQUATION OF X ON Y:

$(X - \bar{X}) = b_{xy} (Y - \bar{Y})$

Where bxy = $\frac{N\sum XY - \sum X \sum Y}{N\sum Y^2 - (\sum Y)^2}$

$$= \frac{8\,(468) - (72)\,(48)}{8\,(336) - (48)^2}$$

$$= \frac{3744 - 3456}{2688 - 2304}$$

$$= \frac{288}{384}$$

= .75

So (X – 9) = .75 (Y – 6) X – 9 = .75Y – 4.5

X = 4.5 + .75Y

REGRESSION EQUATION OF Y ON X:

$(Y - \bar{Y}) = b_{xy} (X - \bar{X})$

Where byx = $\frac{N\sum XY - \sum X \sum Y}{N\sum X^2 - (\sum X)^2}$

$$= \frac{8\,(468) - (72)\,(48)}{8\,(676) - (72)^2}$$

$$= \frac{3744 - 3456}{5408 - 5184}$$

$$= \frac{288}{224}$$

$$= 1.286$$

So $(Y - 6) = 1.286\,(X - 9)$

$Y - 6 = 1.286X - 11.57$

Y = – 5.57+ 1.286X

Example 3. Find out two Regression equations on basis of the data given below:

	X	Y
Mean	60	80
Standard Deviation (S.D.)	16	20
Coefficient of Correlation	.9	

Also find value of X when Y = 150 and value of Y when X = 100.

Solution:

REGRESSION EQUATION OF X ON Y:

$(X - \bar{X}) = b_{xy} (Y - \bar{Y})$ Where

bxy = $r\,\frac{\sigma x}{\sigma y}$

$$= .9\,\frac{16}{20}$$

$$= .72$$

So $(X - 60) = .72\,(Y - 80)$

$X - 60 = .72Y - 57.6$

X = 2.4 + .72Y

When Y = 150 than X = 2.4 + .72(150) = 110.4

REGRESSION EQUATION OF Y ON X:

$(Y - \bar{Y}) = b_{xy} (X - \bar{X})$ Where

byx = $r \frac{\sigma y}{\sigma x}$

$= .9 \frac{20}{16}$

$= 1.125$

So (Y – 80) = 1.125 (X – 60)

Y – 80 = 1.125X – 67.5

Y = 12.5+ 1.125X

When X = 100 than Y = 12.5 + 1.125 (100) = 125

10.12 PROPERTIES OF REGRESSION COEFFICIENTS

The regression coefficients discussed above have a number of properties which are given as under :

1. The geometric Mean of the two regression coefficients gives the coefficients of correlation i.e. r = $\sqrt{bxy \times byx}$
2. Both the regression coefficients must have the same sign i.e. in other words either both coefficients will have + signs or both coefficients will have - signs. This is due to the fact that in first property we have studied that geometric means of both coefficients will give us value of correlation. If one sign will be positive and other will be negative, the product of both signs will be negative. And it is not possible to find out correlation of negative value.
3. The signs of regression coefficients will give us signs of coefficient of correlation. This means if the regression coefficients are positive the correlation coefficient will be positive, and if the regression coefficients are negative then the correlation coefficient will be negative.
4. If one of the regression coefficients is greater than unity or 1, the other must be less than unity. This is because the value of coefficient of correlation must be in between ± 1. If both the regression coefficients are more than 1, then their geometric mean will be more than 1 but the value of correlation cannot exceed 1.

a. 5. The arithmetic mean of the regression coefficients is either equal to or more than the correlation coefficient $\frac{bxy+byx}{2} \geq \sqrt{bxy \times byx}$

5. If the regression coefficients are given we can calculate the value of standard deviation by using the following formula.

a. bxy = $r \frac{\sigma x}{\sigma y}$ or byx = $r \frac{\sigma y}{\sigma x}$

6. Regression coefficients are independent of change of origin but not of scale. This means that if the original values of the two variables are added or subtracted by

some constant, the values of the regression coefficients will remain the same. But if the original values of the two variables are multiplied, or divided by some constant (common factors) the values of the regression equation will not remain the same.

Example 4. From the following data find out two lines of regression land also find out value of correlation.

$\sum X = 250$; $\sum Y = 300$; $\sum XY = 7900$;

$\sum X^2 = 6500$; $\sum Y^2 = 10000$; $n = 10$

Solution:

$$\bar{X} = \frac{\Sigma X}{N} = \frac{250}{10} = 25$$

$$\bar{Y} = \frac{\Sigma Y}{N} = \frac{300}{10} = 30$$

REGRESSION EQUATION OF Y ON X:

$(Y - \bar{Y}) = b_{xy} (X - \bar{X})$ Where

$$byx = \frac{N\Sigma XY - \Sigma X \Sigma Y}{N\Sigma X^2 - (\Sigma X)^2}$$

$$= \frac{10\ (7900) - (250)\ (300)}{10\ (6500) - (250)^2}$$

$$= \frac{79000 - 75000}{65000 - 62500}$$

$$= \frac{4000}{2500}$$

$= 1.6$

So $(Y - 30) = 1.6\ (X - 25)$ $Y - 30 = 1.6X - 40$

Y = – 10+ 1. 6X

REGRESSION EQUATION OF X ON Y:

$(X - \bar{X}) = b_{xy} (Y - \bar{Y})$ Where

$$bxy = \frac{N\Sigma XY - \Sigma X \Sigma Y}{N\Sigma Y^2 - (\Sigma Y)^2}$$

$$= \frac{10\ (7900) - (250)\ (300)}{10\ (10000) - (300)^2}$$

$$= \frac{79000 - 75000}{100000 - 90000}$$

$= \frac{4000}{10000}$

$= .4$

So (X – 25) = .4(Y – 30)

X – 25 = .4Y – 12

X = 13 + .4Y

Coefficients of Correlation

$= \sqrt{bxy \times byx}$

r r $= \sqrt{1.6 \times .4}$ r

$= \sqrt{.64}$ r = .8

Example 5. From the following data find out two lines of regression land also find out value of correlation. Also find value of Y when X = 30

$\sum X = 140$; $\sum Y = 150$; $\sum(X - 10)(Y - 15) = 6$;

$\sum(X - 10)^2 = 180$; $\sum(Y - 15)^2 = 215$; n = 10

Solution:

Lets take assumed mean of Series X = 10 and Series Y = 15.

$\sum dx = \sum(X - 10) = \sum X - 10n = 140 - 100 = 40$

$\sum dy = \sum(Y - 15) = \sum Y - 15n = 150 - 150 = 0$

$\sum dx^2 = \sum(X - 10)^2 = 180$

$\sum dy^2 = \sum(Y - 15)^2 = 215$

$\sum dxdy = \sum(X - 10)(Y - 15) = 6$

So

$\bar{X} = A + \frac{\Sigma X}{N} = 10 + \frac{40}{10} = 14$

$\bar{Y} = A + \frac{\Sigma Y}{N} = 15 + \frac{0}{10} = 15$

REGRESSION EQUATION OF Y ON X:

$(Y - \bar{Y}) = b_{xy}(X - \bar{X})$

Where byx = $\frac{N\sum dxdy - \sum dx \sum dy}{N\sum dx^2 - (\sum dx)^2}$

$= \frac{10(6) - (40)(0)}{10(180) - (40)^2}$

$= \frac{60}{200}$

$= .3$

So (Y – 15) = .3 (X – 14)

Y – 15 = .3X – 4.2

$Y = 10.8 + .3X$

When X = 30 than Y = 10.8 + .3(30) = 19.8

REGRESSION EQUATION OF Y ON X:

$(Y - \bar{Y}) = b_{xy} (X - \bar{X})$

Where byx = $\frac{N\sum dxdy - \sum dx \sum dy}{N\sum dx^2 - (\sum dx)^2}$

$= \frac{10\ (6) - (40)\ (0)}{10\ (25) - (0)^2}$

$= \frac{60}{250}$

= .24

So $(Y - 15) = .24\ (X - 14)$

Y – 15 = .24X – 3.36

$Y = 11.64 + .24X$

Coefficients of Correlation

r = $\sqrt{bxy \times byx}$ r

= $\sqrt{.3 \times .24}$ r = $\sqrt{.072}$

r = .268

Example 5. From the following data find out which equation is equation X on Y and which equation is equation Y on X. Also find $\bar{X}$, $\bar{Y}$ and r.

3X + 2Y – 26 = 0

6X + Y – 31 = 0 **Solution:**

To find $\bar{X}$ and $\bar{Y}$, we will solve following simultaneous equations

3X + 2Y = 26 .... (i)

6X + Y = 31 .... (ii)

Multiply equation (i) with 2, we get

6X + 4Y = 52 .... (iii)

Deduct equation (ii) from equation (iii)

6X + 4Y = 52

-<u>6X - Y = -31</u>

3Y = 21 Y

= 7

Or $\bar{Y}$ = 7.

Put the value of Y in Equation (i), we get

$3X + 2(7) = 26$
$3X + 14 = 26$
$3X = 12$
$X = 4$
or $\bar{X} = 4$
Let $3X + 2Y = 26$ be regression equation X on Y
$3X = 26 - 2Y$

$X = \frac{26}{3} - \frac{2}{3}Y$

So $b_{xy} = -\frac{2}{3}$

Let $6X + Y = 31$ be regression equation Y on X
$Y = 31 - 6X$

So $b_{yx} = -6$

As $r = \sqrt{bxy \times byx}$
$r = -\sqrt{-(\frac{2}{3}) \times (-.6\ )}$
$r = -2$, but this is not possible as value of r always lies between -1 *and* $+1$. So, our assumption is wrong and equation are reverse.
Let $6X + Y = 31$ be regression equation X on Y
$6X = 31 - Y$

$X = \frac{31}{6} - \frac{1}{6}Y$
So $b_{xy} = -\frac{1}{6}$

Let $3X + 2Y = 26$ be regression equation Y on X
$2Y = 26 - 3X$ $Y = \frac{26}{2} - \frac{3}{2}X$

So $b_{yx} = -\frac{3}{2}$

As $r = \sqrt{bxy \times byx}$

$r = -\sqrt{-\left(\frac{1}{6}\right) \times -\left(\frac{3}{2}\right)}$ $r = -.5$, which is possible. So,

our assumption is right.

So,

$\bar{Y}$ = 7;
$\bar{X}$ = 4;
X on Y is X = $\frac{31}{6} - \frac{1}{6}$Y
Y on X is Y = $\frac{26}{2} - \frac{3}{2}$X
r = $-.5$

10.13 TEST YOUR UNDERSTANDING (D)

1. Find both regression equations:

X	6	2	10	4	8
Y	9	11	5	8	7

2. From following estimate the value of Y when X = 30 using regression equation.

X	25	22	28	26	35	20	22	40	20	18	19	25
Y	18	15	20	17	22	14	15	21	15	14	16	17

3. Fit two regression lines:

X	30	32	38	35	40
Y	10	14	16	20	15

Find X when Y = 25 and find Y when X = 36.

4. Find out two Regression equations on basis of the data given below:

	X	Y
Mean	65	67
Standard Deviation (S.D.)	2.5	3.5
Coefficient of Correlation	.8	

5. In a data the Mean values of X and Y are 20 and 45 respectively. Regression coefficient b_{yx} = 4 and b_{xy} 1/9. Find

a. coefficient of correlation
b. Standard Deviation of X, if S.D. of Y = 12
c. Find two regression lines

6. You are supplied with the following information. Variance of X = 36 12X – 51Y + 99 = 0 60X – 27Y = 321.

Calculate

(a) The average values of X and Y

(b) The standard deviation of Y and

7. The lines of regression of Y on X and X on Y are Y = X + 5 and 16X = 9Y + 4 respectively

Also σy=4 Find $\bar{X}$, Y , σx and r.

8. Given :

$$\sum X = 56 , \sum Y = 40 , \sum X^2 = 524$$

$$\sum Y^2 = 256 , \sum XY = 364 , N = 8$$

(a) $find the regression equatiopn of X on Y$

Answers

1. **X = 16.4 – 1.3Y, Y = 11.9 - .65X**
2. **18.875**
3. **Y = .46X – 1.1, X = .6Y + 26, Value of Y = 15.46, Value of X = 40.25**

4. **Y = 1.12X – 5.8, X = .57Y +26,81**
5. **.67, 2, Y= 4X – 35 and X = 1/9 Y +15**
6. **Mean of X = 13, Mean of Y= 17, S.D of Y = 8**
7. **Mean of X = 7, Mean of Y= 12, S.D of X = 3, r= .75,**
8. **X = 1.5Y - 0.5, r = .977**

10.14 LET US SUM UP

- **Regression is an useful tool of forecasting.**
- **With help of regression we can predict the value of can find the value of X if value of Y is given or value of Y if value of X is given.**
- **It creates the mathematical linear relation between two variables X and Y, out of which one variable is dependent and other is independent.**
- **In this we find out two regression equations.**
- **Regression can be linear or non linear.**
- **It can be simple or multiple.**
- **Regression is based on the principle of Least Squares. • We can also find out correlation coefficient with help of regression coefficients.**

10.15 KEY TERMS

- **REGRESSION:** Regression createsthe mathematical linear relation between two variables X and Y, out of which one variable is dependent and other is independent.
- **SIMPLE REGRESSION**: When there are only two variables under study it is known as a simple regression. For example we are studying the relation between Sales and Advertising expenditure.
- **MULTIPLE REGRESSION:** The study of more than two variables at a time is known as multiple regression. Under this, only one variable is taken as a dependent variable and all the other variables are taken as independent variables.
- **TOTAL REGRESSION:** Total regression analysis is one in which we study the effect of all the variables simultaneously.
- **PARTIAL REGRESSION:** In the case of Partial Regression one or two variables are taken into consideration and the others are excluded.
- **LINEAR REGRESSION:**When the functional relationship between X and Y is expressed as the first degree equations, it is known as linear regression. In other words,

when the points plotted on a scatter diagram concentrate around a straight line it is the case of linear regression.

- **NON-LINEAR REGRESSION**: On the other hand if the line of regression (in scatter diagram) is not a straight line, the regression is termed as curved or non-linear regression.
- **LEAST SQUARE METHOD**: According to the Least Square method, regression line should be plotted in such a way that sum of square of the difference between actual value and estimated value of the dependent variable should be least or minimum possible.

10.16 REVIEW QUESTIONS

18. What is Regression. What are uses of Regression.
19. What is relation between Regression and correlation.
20. Explain different types of regressions.
21. How two regression lines are determined under direct method.
22. Explain various methods of finding regression equations.
23. What are limitations of regression analsis.
24. What are properties of regression coefficients.

10.17 FURTHER READINGS

1. J. K. Sharma, *Business Statistics*, Pearson Education.
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4. Richard Levin and David S. Rubin, *Statistics for Management*, Prentice Hall of India, New Delhi.HillPublishing Co.

B. COM (DIGITAL)
SEMESTER II

COURSE: BUSINESS MATHEMATICS AND STATISTICS

UNIT 11 – INDEX NUMBERS

STRUCTURE

11.0 OBJECTIVES

After studying the Unit, students will be able to

- Define what are index numbers.
- Describe the uses of Index numbers.
- Understand how index numbers are prepared.
- Understand uses of Index numbers.
- Apply your knowledge in construction of simple index numbers. • Familiarize yourself with the concept of Consumer Price Index numbers.

11.1 INTRODUCTION

Human life is dynamic and hardly there is anything which remains same over a period of time. whether it is price of goods, Population of the country, Industrial Production, Import and Exports of the country, everything changes with the passage of time. It is the tendency of human that he wants to measure the changes that are taking place over a period of time. Now he question arise that how we can measure these changes that are taking place. Index number is one such a statistical tool which can help us in measuring these changes.

11.2 HISTORY OF INDEX NUMBERS

An index number is a statistical tool that measure the changes in the data over the period of time. Index number is not a new tool used in statistics, rather the use of index numbers is very old. As per available records, index number was first time constructed in the year 1764 by an Italian named Carli. In his index number, Carli compared the prices of the Year 1750 with the price level of the year 1500. Though normally index numbers are used for measuring the change in price over a period of time, but hardly there is any area in Economics or Commerce where Index numbers are not used. Tere are different types of index numbers that are used in economics such as Industrial Production Index , Agricultural Production Index and Population Index etc.

11.3 MEANING OF INDEX NUMBER

An index number is a device with help of which we can measure the relative change in one variable over a period of time. Normally while preparing the index number, we compare the current prices of a product with the price of some past period known as base year. The index number of the base year is mostly taken as 100. Few definitions of index numbers given by different experts are as follows:

According to Croxton and Cowden, "Index numbers are devices for measuring differences in the magnitude of a group of related variables."

According to Maslow,"Index number is a numerical value characterising the changes in complex economic phenomenon over a period of time or space."

According to A.L. Bowley, "A series of index numbers reflects in its trend and fluctuations the movements of some quantity to which it is related."

According to Spiegel, "An Index number is a statistical measure designed to show changes in a variable, or a group of related variables with respect to time, geographic location, or other characteristics such as income, profession etc."

11.4 FEATURES OF INDEX NUMBER

1. Index numbers are specialised type of average. Normally used measures of average like Mean Median and Mode can be used for two or more different series, if their units are same. In case units of two series are different, these cannot be represented by normal average, However, Index number can help in this situation.
2. Normally index numbers are represented in percentage. However, the % sign is not used while showing index numbers.
3. Index numbers gives the effect of change over a period of time or the change that is taking place in two different locations.
4. Index numbers measure that change which is not capable of measurement normally in quantitative figures. For example we cannot measure the change in cost of living directly, but Index numbers can help us in this situation.

11.5 USES OF INDEX NUMBERS

1. Index number is a very powerful tool of economic and business analysis. We often call index number as ‘Barometer of the Economy’. With the help of Index Number we can see pulse of the economy.
2. Index number is a very helpful tool in planning of activities and formulation of business policy.

3. With the help of index numbers, economists try to find out trends in prices, production, import and exports etc.
4. Index number shows the cost of living over a period of time. This also helps government in fixing the wage rate of the labour. 5. Index number also helps us in calculation of Real National Income of the country.

11.6 LIMITATIONS OF INDEX NUMBERS

1. As index numbers are based on sample data, these can give only approximate result not the accurate result.
2. Index number normally deals with one variable, so it is not possible to calculate a single index for all the economic activities.
3. There is not a single standard method of calculating index. Different experts calculate index in their own way.
4. Index number are special types of average, it does not deal with all the situations. 5. Finding the appropriate base period is very difficult in construction of index number.

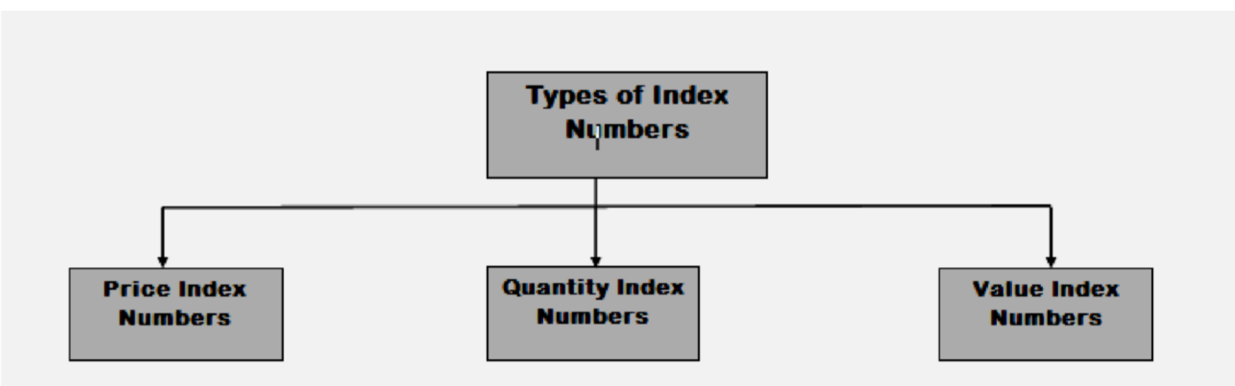
11.7 PROBLEMS IN CONSTRUCTION OF INDEX NUMBERS

1. **PURPOSE OF INDEX NUMBERS**: The first step in construction of index number is to decide the purpose of preparing the Index Number. As there is no single purpose index, so we must decide the objective of index very carefully.
2. **SELECTION OF BASE YEAR**: The selection of base period is most important step in preparation of index number. Best base period is the period with which we can find accurate change in the variable. Following are some of the guidelines that must be kept in mind while selecting the base period.
   - The period selected as base period should be normal one. There must not be any problem like War, Flood, Earthquake, Economic Depression etc. in the base period.
     • The difference between base period and the current period should not be very large
   - Only that period should be taken as base period full data is available.
3. **SELECTION OF NUMBER OF ITEMS OR COMMODITIES**: The next major problem in preparation of index number is to select the number of items that will form the Index number. Following points must be kept in mind while deciding the number of items in the index numbers.
   - Only those items should be selected that represent the habits and taste of majority of customers.
   - The number of items selected should not be very large or very small.
   - Only those items should be selected that are available in standard quantity.

- Only those items must be selected that were available in base period as well as current period.

4. **SELECTION OF SOURCE OF DATA:** As in index numbers, we compare current variables with the variables of past periods, the source from which data is collected must be authentic one. In case of non authentic data, it will give wrong picture.
5. **PRICE QUOTATIONS:** The prices of the commodities differs from place to place, It is very important to select that price which represent majority of places. Further while preparing index number we may take wholesale prices or retail prices in consideration.
6. **SELECTION OF THE AVERAGE**: There are different types of averages, like Arithmetic Mean, Geometric Mean, Harmonic Mean, Median and Mode that can be used in preparation of index number. ee must select appropriate average in preparation of index number based on our objective.
7. **SELECTION OF APPROPRIATE WEIGHT:** The next major problem in preparation of index number is to assign weight to the different items. All the items of the data under consideration are not equally important, some items may be more important and some items may be less important. So, more weight must be assigned to important items while preparing the index number. Now the problem is that how to assign weights to the items. Normally we take quantity of the items consumed as weight in Index Number.
8. **SELECTION OF APPROPRIATE FORMULA:** There are a number of formula that can be used for preparing index numbers. for example Laspeyer's method. Bowle Method and Fisher Method etc. Each method has its own advantages and limitations. so must be selected very carefully.

11.8 DIFFERENT TYPES OF INDEX NUMBERS



1. **PRICE INDEX NUMBERS**: These index numbers are used for measuring the change in prices of the commodities over a period of time. In other words we can say that these index numbers find the change in value of money over a period of time. These index numbers are most popular index number. These Index numbers may be based on Wholesale Price Index or Retail Price Index.
2. **QUANTITY INDEX NUMBERS**: The Quantity or Volume Index Numbers measure the change in quantities used by people over a period of time. under these index numbers,

we calculate change in physical quantity of goods produced, consumed or sold over a period of time. There are different types of quantity index numbers such as Agricultural Production Index Number, Industrial Production Index Number, Export Import Index Number etc.

3. **VALUE INDEX NUMBERS**: Value Index Numbers compare the change in total value over period of time. These index numbers takes into consideration both prices and quantity of the product while finding the change over a period of time. These Index Numbers are very useful in finding consumption habits of the consumers.

11.9 DIFFERENT METHODS OF INDEX NUMBERS

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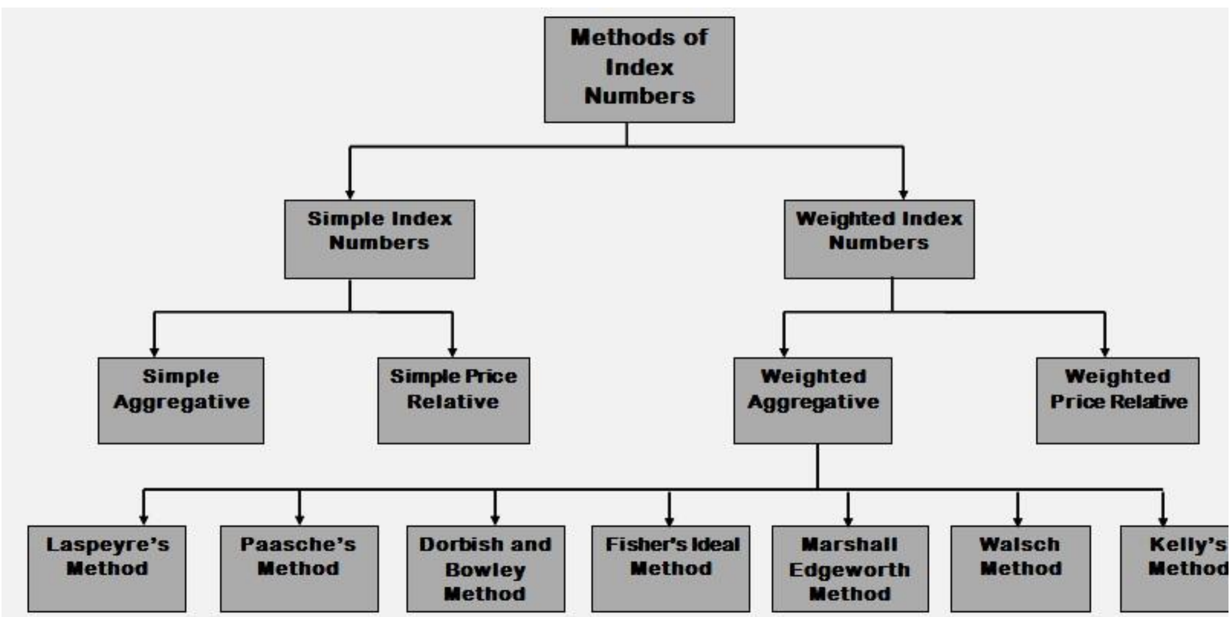
As we have already discussed, Index number is a device that shows that changes in price over a f period of time. Now a question arise that how to calculate the index number. The re are a number of methods for preparing the index numbers. Following chart shows various methods preparing index numbers.



11.10 SIMPLE AGGREGATIVE METHOD

This is one of the old and simple method of finding the index number. Under this method we calculate the index number of a given period by dividing the aggregate of all the prices of the current year with the the aggregate of all the prices of the base year. After that we multiply the resultant figure with 100 to find the index number. Following are the steps:

1. Decide the base year.

2. Add all the prices of base year for all available comodities, it is denoted by $\sum P_0$.
3. Add all the prices of base year for all available comodities, it is denoted by $\sum P_1$.
4. Use following the formula for calculating index number under this method:

$$P_{01} = \frac{\sum P_1}{\sum P_0} \times 100$$

Where

P01 – Price Index Number of Current Year

$\sum P_1$ – Aggregate of Prices of Current Year

$\sum P_0$ – Aggregate of Prices of Base Year

Example 1. Construct Simple Aggregative Index number of the year 2020by taking the base as prices of 2015.

Commodity	**Price of the Year 2015**	**Price of the Year 2020**
Wheat	**20**	**26**
Sugar	**40**	**34**
Oil	**60**	**120**
Pulses	**80**	**140**

Solution:

Price Index (Year 2015 taken as the base year)

Commodity	Price of the Year 2015 P_0	Price of the Year 2020 P_1
Wheat	20	26
Sugar	40	34
Oil	60	120
Pulses	80	140
	$\sum P_0 = 200$	$\sum P_1 = 320$

Price Index (P_{01}) $= \frac{\sum P1}{\sum P0} \times 100 = \frac{320}{200} \times 100 = 160$

Price index shows that prices have increased by 60% in 2020 than 2015.

11.10.1 MERITS OF SIMPLE AGGREGATIVE METHOD

1. This method is simple to calculate.
2. This method is very simple to understand.
3. This method does not need much mathematical calculations

11.9.2 LIMITATIONS OF SIMPLE AGGREGATIVE METHOD

1. This method does not give change in price over a period of time.
2. Prices of different commodities are measured in different units like some are measured in Kilograms where as other in Meters etc. It creates problems in calculation.
3. This method is influenced by unit of measurement.

4. This method ignore the relative importance of the item.
5. This method use only Arithmetic mean as a tool for calculating index number. Other measures of average like Geometric mean or median etc. cannot be used in this method.
6. Index number in this method is influenced by magnitude of the price.

11.11 SIMPLE PRICE RELATIVE METHOD

This method is a bit improvement over the simple aggregative method. Simple aggregative method is affected by the magnitude of the price of the item. However, this method is not affected by magnitude of the price of item. Further, in this method it is not necessary to use Arithmetic mean as average rather we can use any method of finding average, such as Arithmetic mean, Geometric mean, Median, Mode etc. However, normally we prefer to use Arithmetic mean in this case. Following are the steps of this method:

1. Decide the base year.
2. Calculate the price relative of current year for each commodity by dividing current Prices (P_1) with base year price (P_0) using the following formula $\frac{P1}{P_0}. \times 100$
3. Find sum of all the price relatives so calculated.
4. Divide the sum or price relatives with number of items to get index number by using the following formula:

$$P_{01} = \frac{\sum \frac{P_1}{P_0} \times 100}{N}$$

11.11.1 MERITS OF SIMPLE PRICE RELATIVE METHOD:

1. This method is very simple to calculate and understand.
2. This method is not affected by the magnitude of price of a particular item.
3. This method is not affected by unit of measurement of the item.
4. This method is not necessarily based on Arithmetic Mean, we can use other averages like Geometric Mean, median etc also. 5. Equal weights are provided to each item.

11.11.2 LIMITATIONS OF SIMPLE PRICE RELATIVE METHOD:

1. Selection of average is a difficult task in this method.
2. If it is to be calculated using Geometric Mean, than calculation is very difficult. 3. It does not consider which item is used more and provide equal weights to all items.

Example 2. Construct Simple Price Relative Index number of the year 2020by taking the base as prices of 2015.

Commodity	Price of the Year 2015	Price of the Year 2020
Wheat	20	26
Sugar	40	34
Oil	60	120

Pulses	**80**	**140**

Solution:

Price Index (Year 2015 taken as the base year)

Commodity	Price of the Year 2015 P_0	Price of the Year 2020 P_1	Price Relative $\frac{P_1}{P_0} \times 100$
Wheat	20	26	$\frac{26}{20} \times 100 = 130$
Sugar	40	34	$\frac{34}{40} \times 100 = 85$
Oil	60	120	$\frac{120}{60} \times 100 = 200$
Pulses	80	140	$\frac{140}{80} \times 100 = 175$
			$\sum\frac{P_1}{P_0} \times 100 = 590$

Price Index (P_{01}) = $\frac{\sum\frac{P_1}{P_0}\times 100}{N} = \frac{590}{4} = 147.50$

Price index shows that prices have increased by 47.5% in 2020 than 2015.

11.12 TEST YOUR UNDERSTANDING (A)

1. Calculate Index number for 2015 taking 2011 as base using Simple Aggregative Method and Simple Average of Relatives Method:

Items	Price 2011	Price 2015
A	350	510
B	45	40
C	77	156
D	37	47
E	10	12

2. Find index using simple average of price relative using 2017 as base.

Items	Price 2017	Price 2019
A	15	30
B	18	24
C	16	20

D	14	21
E	25	35
F	40	30

3. Find simple aggregative index

Items	P_0	P_1
Oil	60	70
Pulses	70	60
Rice	50	40
Sugar	40	40

Answers

1. 147.4, 132.84

2. 137.22

3. 95.45

11.13 WEIGHTED AGGREGATIVE PRICE INDEX

Simple Aggregative methods of Index Numbers assume that all the items of Index Number are equally important. There is no item which is more important than other. So, this method provide equal weightage to all items. However, in practical life it is not true. Some items carry more importance than other items, for example in human's life expenditure on food carries more importance than expenditure on entertainment. So, we have weighted method of index numbers which considers relative importance of the item also.

Weighted Aggregative Method is one such method. This method is more or less same as Simple Aggregative Method but main difference is that is also considers relative weights of the items also. Generally the quantity of the item consumed is considered as weight in this case. There are many methods of calculating Weighted Aggregative Price Index which are discussed as follows:

11.13.1 LASPEYRE'S METHOD:

This method was suggested by Mr. Laspeyre in 1871. Under this method base year quantities of the various products are assumed as weight for preparing the index numbers. The following steps may be used:

1. Multiply Prices of the base year (P_0) with the quantities of the base year (Q_0) for every commodity.
2. Add the values calculated in step 1, the sum is denoted as $\sum P_0 Q_0$
3. Multiply Prices of the current year (P_1) with the quantities of the base year (Q_0) for every commodity.
4. Add the values calculated in step 3, the sum is denoted as $\sum P_1 Q_0$.
5. Use following formula for calculating index number:

$$P_{01} = \frac{\sum P_1Q_0}{\sum P_0Q_0} \times 100$$

11.13.2 PAASCHE'S METHOD:

This method was suggested by Mr. Paasche in 1874. Under this method current year quantities of the various products are assumed as weight for preparing the index numbers. The following steps may be used:

1. Multiply Prices of the base year (P_0) with the quantities of the current year (Q_1) for every commodity.
2. Add the values calculated in step 1, the sum is denoted as $\sum P_0 Q_1$
3. Multiply Prices of the current year (P_1) with the quantities of the current year (Q_1) for every commodity.
4. Add the values calculated in step 3, the sum is denoted as $\sum P_1 Q_1$.
5. Use following formula for calculating index number:

$$P_{01} = \frac{\sum P_1Q_1}{\sum P_0Q_1} \times 100$$

11.13.3 DORBISH AND BOWLEY'S METHOD:

This method is based on both Lasypeyre's Method and Paasche's Method, that's why this method is also known as L-P formula. Under this method we calculate the index number by taking the arithmetic mean of the formula given by Laspeyre and Paasche. So, following formula is used in case of the Dorbish and Bowley Method:

$$P_{01} = \frac{\frac{\sum P_1Q_0}{\sum P_0Q_0} + \frac{\sum P_1Q_1}{\sum P_0Q_1}}{2} \times 100$$

11.13.4 FISHER'S IDEAL INDEX METHOD:

This method was suggested by Prof Irving Fisher and it is assumed as one of the best method of constructing the Index Number. That's why this method is also called Ideal Index Number. This method is based on both Lasypeyre's Method and Paasche's Method, but instead of taking arithmetic mean of the both formulas, Fisher used the geometric mean on the formula given by Laspeyre and Paasche. So, following formula for calculating Fisher's ideal Index number:

$$\sqrt{\frac{\sum P_1Q_0}{\sum P_0Q_0} \times \frac{\sum P_1Q_1}{\sum P_0Q_1}} \times 100$$

Fisher's Method is called ideal index number due to following reasons:

1. This method use geometric mean as base which is perhaps best average for constructing Index numbers.

2. This method considers both quantities of base year as well as current year as weight.
3. This method satisfies both the time reversal and factor reversal test. 4. It is comprehensive method and cover all values of data i.e P_0, Q_0, P_1, Q_1 etc.

11.13.5 MARSHAL EDGEWORTH INDEX METHOD:

Like the Fisher's method, this method also use the quantities of base as well as current year as weight. Under this method arithmetic mean of the quantity of base and current year is assumed as weight. This method is comparatively simple than Fisher's method as it does not use complex concept of Geometric mean. Following is the formula of this method.

$$P_{01} = \frac{\sum P_1(Q_0 + Q_1)}{\sum P_0(Q_0 + Q_1)} \times 100 \text{ or}$$

$$\frac{\sum P_1Q_0 + \sum P_1Q_1}{\sum P_0Q_0 + \sum P_0Q_1} \times 100$$

11.13.6 WALSCH INDEX METHOD:

This method also use the quantities of base as well as current year as weight. Under this method Geometric mean of the quantity of base and current year is assumed as weight. Following is the formula of this method.

$$P_{01} = \frac{\sum P_1\sqrt{Q_0 \times Q_1}}{\sum P_0\sqrt{Q_0 \times Q_1}} \times 100$$

11.13.7 KELLY'S INDEX METHOD:

This method was contributed by Truman Kelly. Under this method particular weights are fixed for each period and that are uses in index number. That's why this method is also called aggregative index with fixed weighs. Following formula is used in this method:

$$P_{01} = \frac{\sum P_1Q}{\sum P_0Q} \times 100$$

Example 3. Construct Weighted Aggregative Index number of the year 2020 by taking the base as prices of 2015 using Laspeyre, Paasche, Dorbish & Bowley, Fisher, Marshal Edgeworth and Kelly's method.

Item	Price of the Year 2015	Quantity of the Year 2015	Price of the Year 2020	Quantity of the Year 2020
A	6	50	10	56
B	2	100	2	120
C	4	60	6	60
D	10	30	12	24

E	**8**	**40**	**12**	**36**

Solution:

Item	P_0	Q_0	P_1	Q_1	P_0Q_0	P_0Q_1	P_1Q_0	P_1Q_1
A	6	50	10	56	300	336	500	560
B	2	100	2	120	200	240	200	240
C	4	60	6	60	240	240	360	360
D	10	30	12	24	300	240	360	288
E	8	40	12	36	320	288	480	432
					$\sum P_0Q_0$ = 1360	**$\sum P_0Q_1$ = 1344**	**$\sum P_1Q_0$ = 1900**	**$\sum P_1Q_1$ = 1880**

1. Laspeyre's Method:

$$P_{01} = \frac{\sum P_1Q_0}{\sum P_0Q_0} \times 100 = \frac{1900}{1360} \times 100 = 139.71$$

2. Paasche's Method:

$$P_{01} \frac{\sum P_1Q_1}{\sum P_0Q_1} \times 100 = \frac{1880}{1344} \times 100 = 139.88$$

3. Dorbish and Bowley's Method:

$$P_{01} = \frac{\frac{\sum P_1Q_0}{\sum P_0Q_0} + \frac{\sum P_1Q_1}{\sum P_0Q_1}}{2} \times 100$$

$$= \frac{\frac{1900}{1360} + \frac{1880}{1344}}{2} \times 100 = \frac{2.796}{2} = 139.79$$

4. Fisher's Ideal Index Method:

$$\sqrt{\frac{\sum P_1Q_0}{\sum P_0Q_0} \times \frac{\sum P_1Q_1}{\sum P_0Q_1}} \times 100$$

$$= \sqrt{\frac{1900}{1360} \times \frac{1880}{1344}} \times 100 = \sqrt{1.9543} \times 100 = 139.79$$

5. Marshal Edgeworth Index Method:

$$\frac{\sum P_1Q_0 + \sum P_1Q_1}{\sum P_0Q_0 + \sum P_0Q_1} \times 100$$

$$= \frac{1900 + 1880}{1360 + 1344} \times 100 = \frac{3780}{2704} \times 100 = 139\,.79$$

6. Walsch Index Method:

Item	P_0	Q_0	P_1	Q_1	$\sqrt{O_0 \times O_1}$	$P_1\sqrt{O_0 \times O_1}$	$P_0\sqrt{O_0 \times O_1}$
A	6	50	10	56	52.9	529	317.4
B	2	100	2	120	109.5	219	219.0
C	4	60	6	60	60.0	360	240.0
D	10	30	12	24	26.8	321.6	268.0
E	8	40	12	36	37.9	454.8	303.2
						$\sum P_1\sqrt{O_0 \times O_1}$ **= 1884.8**	$\sum P_0\sqrt{O_0 \times O_1}$ **= 1347.6**

$$P_{01} = \frac{\sum P_1\sqrt{Q_0 \times Q_1}}{\sum P_0\sqrt{Q_0 \times Q_1}} \times 100 = \frac{1884.8}{1347.6} \times 100 = 139.08$$

Example 4. Construct Weighted Aggregative Index number using Laspeyre, Paasche, Dorbish & Bowley and Fisher, method.

Item	**Price of the Base Year**	**Expenditure of the Base Year**	**Price of the Current Year**	**Expenditure of the Current Year**
A	**2**	**40**	**5**	**75**
B	**4**	**16**	**8**	**40**
C	**1**	**10**	**2**	**24**
D	**5**	**25**	**10**	**60**

Solution:

We know that Expenditure = Price × Quantity

So Quantity $= \frac{Expenditure}{Price}$

Item	P_0	Q_0	P_1	Q_1	P_0Q_0	P_0Q_1	P_1Q_0	P_1Q_1
A	2	20	5	15	40	30	100	75
B	4	4	8	5	16	20	32	40
C	1	10	2	12	10	12	20	24
D	5	5	10	6	25	30	50	60
					$\sum P_0Q_0$ = 91	**$\sum P_0Q_1$ = 92**	**$\sum P_1Q_0$ = 202**	**$\sum P_1Q_1$ = 199**

1. Laspeyre's Method:

$$P_{01} = \frac{\sum P_1Q_0}{\sum P_0Q_0} \times 100 = \frac{202}{91} \times 100 = 221.98$$

2. Paasche's Method:

$$P_{01} \frac{\sum P_1Q_1}{\sum P_0Q_1} \times 100 = \frac{199}{92} \times 100 = 216.39$$

3. Dorbish and Bowley's Method:

$$P_{01} = \frac{\frac{\sum P_1Q_0}{\sum P_0Q_0} + \frac{\sum P_1Q_1}{\sum P_0Q_1}}{2} \times 100$$

$$= \frac{\frac{202}{91} + \frac{199}{92}}{2} \times 100 = \frac{4.3828}{2} = 219.14$$

4. Fisher's Ideal Index Method:

$$\sqrt{\frac{\sum P_1Q_0}{\sum P_0Q_0} \times \frac{\sum P_1Q_1}{\sum P_0Q_1}} \times 100$$

$$= \sqrt{\frac{202}{191} \times \frac{199}{92}} \times 100 = \sqrt{4.8015} \times 100 = 219.12$$

11.14 WEIGHTED PRICE RELATIVE METHOD

This method almost similar to simple price relative method. However, simple price relative give equal importance to all items under consideration. But in our life, all items do not carry equal importance. Some item are more important or on some items we spend more amount. Change in price of some items affect us more than change in price of some other items. So, we have weighted price relative method. This method is similar to simple price relative method but also assigns weights to the items. Further, in this method it is not necessary to use Arithmetic mean as average rather we can use any method of finding average, such as Arithmetic mean, Geometric mean, etc. However, normally we prefer to use Arithmetic mean in this case.

Following are the steps of this method:

1. Decide the base year.
2. Calculate the price relative of current year for each commodity by dividing current Prices (P_1) with base year price (P_0) using the following formula $\frac{P1}{P_0} \times 100$.
3. Find the weights of the items to be assigned.
4. Multiply price relative so calculated with the weights and find out the product of both.
5. Find sum of product so calculated.
6. Find sum of the weights assigned.
7. Divide the sum of the weighted price relatives with sum of weights to get index number by using the following formula:

$$P_{01} = \frac{\sum W \frac{P_1}{P_0} \times 100}{\sum W}$$

11.14.1 MERITS OF WEIGHTED PRICE RELATIVE METHOD:

1. This method is very simple to calculate and understand.
2. This method is not affected by the magnitude of price of a particular item.
3. This method is not affected by unit of measurement of the item.
4. This method is not necessarily based on Arithmetic Mean, we can use other averages like Geometric Mean, median etc also. 5. Weights are assigned according to importance of the items.

11.14.2 LIMITATIONS OF WEIGHTED PRICE RELATIVE METHOD:

1. Selection of average is a difficult task in this method.
2. If it is to be calculated using Geometric Mean, than calculation is very difficult.
3. Selection of weights is a difficult task.

Example 5. Construct Weighted Price Relative Index number of the year 2020by taking the base as prices of 2015.

Commodity	Price of the Year 2015	Price of the Year 2020	Weights
Wheat	**20**	**26**	**40**
Sugar	**40**	**34**	**5**
Oil	**60**	**120**	**3**
Pulses	**80**	**140**	**2**

Solution:

Price Index (Year 2015 taken as the base year)

Commodity	Price of the Year 2015 P_0	Price of the Year 2020 P_1	Price Relative $\frac{P_1}{P_0} \times 100$	Weights (W)	Weighted Price Relatives $W \frac{P_1}{P_0} \times 100$
Wheat	20	26	$\frac{26}{20} \times 100 = 130$	40	5200
Sugar	40	34	$\frac{34}{40} \times 100 = 85$	5	424
Oil	60	120	$\frac{120}{60} \times 100 = 200$	3	600
Pulses	80	140	$\frac{140}{80} \times 100 = 175$	2	350
				$\sum W = 50$	$\sum W \frac{P_1}{P_0} \times 100 = 6575$

Price Index (P_{01}) = $\frac{\sum W \frac{P_1}{P_0} \times 100}{\sum W} = \frac{6575}{50} = 131.50$

Price index shows that prices have increased by 31.5% in 2020 than 2015.

11.15 TESTS OF CONSISTENCY FOR INDEX NUMBERS

There are a number of methods through which index numbers can be calculated. Each method has its own merits and demerits. Now a question is that which of these method can be treated as best. In order to find out which method is better than others, there are four tests. If any index number satisfies these test we may consider the index number to be ideal one.

11.15.1 UNIT TEST:

Unit test says that any index number can be treated as ideal only if it is free from the unit in which quantity is measured. Whether prices are quoted for single item or for dozen items, the index number must not be affected by the same. Only simple average of price relative method satisfies this condition.

11.15.2 TIME REVERSAL TEST

This test was suggested by Fisher. According to this test, an ideal index number is one which works both ways that i.e. backward and forward. So if index is prepared by taking old period as base year and new period as current year it comes to be 200 it means prices in current period are doubled. Now say reverse is done, new period is taken as base and old period is taken as current year, this test says that index should be 50 which means earlier prices were half of current prices. In other words we can say that following condition should be satisfied

$$\boxed{P_{01} \times P_{10} = 1}$$

Following is the formula of time reversal test in different cases:

1. LASPEYRE'S METHOD:

$$P_{01} \times P_{10} = \frac{\sum P_1Q_0}{\sum P_0Q_0} \times \frac{\sum P_0Q_1}{\sum P_1Q_1} \neq 1$$

This method does not satisfy time reversal test.

2. PAASCHE'S METHOD:

$$P_{01} \times P_{10} = \frac{\sum P_1Q_1}{\sum P_0Q_1} \times \frac{\sum P_0Q_0}{\sum P_1Q_0} \neq 1$$

This method does not satisfy time reversal test.

3. DORBISH AND BOWLEY'S METHOD:

$$P_{01} \times P_{10} = \frac{\frac{\sum P_1Q_0}{\sum P_0Q_0} + \frac{\sum P_1Q_1}{\sum P_0Q_1}}{2} \times \frac{\frac{\sum P_0Q_1}{\sum P_1Q_1} + \frac{\sum P_0Q_0}{\sum P_1Q_0}}{2} \neq 1$$

This method does not satisfy time reversal test.

4. FISHER'S IDEAL INDEX METHOD:

$$P_{01} \times P_{10} = \sqrt{\frac{\sum P_1Q_0}{\sum P_0Q_0} \times \frac{\sum P_1Q_1}{\sum P_0Q_1}} \times \sqrt{\frac{\sum P_0Q_1}{\sum P_1Q_1} \times \frac{\sum P_0Q_0}{\sum P_1Q_0}} = 1$$

This method satisfies time reversal test.

5. MARSHAL EDGEWORTH INDEX METHOD:

$$P_{01} \times P_{10} = \frac{\sum P_1Q_0 + \sum P_1Q_1}{\sum P_0Q_0 + \sum P_0Q_1} \times \frac{\sum P_0Q_1 + \sum P_0Q_0}{\sum P_1Q_1 + \sum P_1Q_0} = 1$$

This method satisfies time reversal test.

6. WALSCH INDEX METHOD:

$$P_{01} \times P_{10} = \frac{\sum P_1\sqrt{Q_0 \times Q_1}}{\sum P_0\sqrt{Q_0 \times Q_1}} \times \frac{\sum P_0\sqrt{Q_1 \times Q_0}}{\sum P_1\sqrt{Q_1 \times Q_0}} = 1$$

This method satisfies time reversal test.

11.15.3 Factor Reversal Test (F.R.T.)

This test was also suggested by Fisher. According to this test, an ideal index number is one which does not give inconsistent result if we change price with quantity and quantity with price. According to this test when we multiply change in price with change in quantity the ratio must be equal to total change in value.

$$\mathbf{P_{01} \times Q_{01}} = \frac{\sum P_1Q_1}{\sum P_0Q_0}$$

Following is the formula of factor reversal test in different cases:

1. LASPEYRE'S METHOD:

$$P_{01} \times Q_{10} = \frac{\sum P_1Q_0}{\sum P_0Q_0} \times \frac{\sum Q_1P_0}{\sum Q_0P_0} \neq \frac{\sum P_1Q_1}{\sum P_0Q_0}$$

This method does not satisfy factor reversal test.

2. PAASCHE'S METHOD:

$$P_{01} \times Q_{10} = \frac{\sum P_1Q_1}{\sum P_0Q_1} \times \frac{\sum Q_1P_1}{\sum Q_0P_1} \neq \frac{\sum P_1Q_1}{\sum P_0Q_0}$$

This method does not satisfy factor reversal test.

3. DORBISH AND BOWLEY'S METHOD:

$$P_{01} \times Q_{10} = \frac{\frac{\sum P_1Q_0}{\sum P_0Q_0} + \frac{\sum P_1Q_1}{\sum P_0Q_1}}{2} \times \frac{\frac{\sum Q_1P_0}{\sum Q_0P_0} + \frac{\sum Q_1P_1}{\sum Q_0P_1}}{2} \neq \frac{\sum P_1Q_1}{\sum P_0Q_0}$$

This method does not satisfy factor reversal test.

4. FISHER'S IDEAL INDEX METHOD:

$$P_{01} \times Q_{10} = \sqrt{\frac{\sum P_1Q_0}{\sum P_0Q_0} \times \frac{\sum P_1Q_1}{\sum P_0Q_1}} \times \sqrt{\frac{\sum Q_1P_0}{\sum Q_0P_0} \times \frac{\sum Q_1P_1}{\sum Q_0P_1}} = \frac{\sum P_1Q_1}{\sum P_0Q_0}$$

This method satisfies factor reversal test.

5. MARSHAL EDGEWORTH INDEX METHOD:

$$P_{01} \times Q_{10} = \frac{\sum P_1Q_0 + \sum P_1Q_1}{\sum P_0Q_0 + \sum P_0Q_1} \times \frac{\sum Q_1P_0 + \sum Q_1P_1}{\sum Q_0P_0 + \sum Q_0P_1} = \frac{\sum P_1Q_1}{\sum P_0Q_0}$$

This method does not satisfy factor reversal test.

6. WALSCH INDEX METHOD:

$$P_{01} \times Q_{10} = \frac{\sum P_1\sqrt{Q_0 \times Q_1}}{\sum P_0\sqrt{Q_0 \times Q_1}} \times \frac{\sum Q_1\sqrt{P_0 \times P_1}}{\sum Q_0\sqrt{P_0 \times P_1}} = \frac{\sum P_1Q_1}{\sum P_0Q_0}$$

This method does not satisfy factor reversal test.

Example 6. Construct Weighted Aggregative Index number using Laspeyre, Paasche, and Fisher method also check whether these satisfy T.R.T. and F.R.T or not

Item	**Price of the Base Year**	**Qty. of the Base Year**	**Price of the Current Year**	**Qty. of the Current Year**
A	**30**	**7**	**40**	**5**
B	**40**	**12**	**60**	**8**
C	**60**	**10**	**50**	**15**
D	**30**	**15**	**20**	**18**

Solution:

We know that Expenditure = Price × Quantity

So Quantity $= \frac{Expenditure}{Price}$

Item	**P_0**	**Q_0**	**P_1**	**Q_1**	**P_0Q_0**	**P_0Q_1**	**P_1Q_0**	**P_1Q_1**
A	30	7	40	5	210	150	280	200
B	40	12	60	8	480	320	720	480
C	60	10	50	15	600	900	500	750
D	30	15	20	18	450	540	300	360
					$\sum P_0Q_0$ = 1740	**$\sum P_0Q_1$ = 1910**	**$\sum P_1Q_0$ = 1800**	**$\sum P_1Q_1$ = 1790**

1. LASPEYRE'S METHOD:

$$P_{01} = \frac{\sum P_1Q_0}{\sum P_0Q_0} \times 100 = \frac{1800}{1740} \times 100 = 103.45$$

Time Reversal Test

$$\frac{\sum P_1Q_0}{\sum P_0Q_0} \times \frac{\sum P_0Q_1}{\sum P_1Q_1} = \frac{1800}{1740} \times \frac{1910}{1790} \neq 1$$

It does not satisfies time reversal test.

Factor Reversal Test

$$\frac{\sum P_1Q_0}{\sum P_0Q_0} \times \frac{\sum Q_1P_0}{\sum Q_0P_0} = \frac{1800}{1740} \times \frac{1910}{1740} \neq \frac{1790}{1740}$$

It does not satisfies facto reversal test.

2. PAASCHE'S METHOD:

$$P_{01} = \frac{\sum P_1Q_1}{\sum P_0Q_1} \times 100 = \frac{1790}{1910} \times 100 = 93.72$$

Time Reversal Test

$$= \frac{\sum P_1Q_1}{\sum P_0Q_1} \times \frac{\sum P_0Q_0}{\sum P_1Q_0} = \frac{1790}{1910} \times \frac{1740}{1800} \neq 1$$

It does not satisfies time reversal test.

Factor Reversal Test

$$\frac{\sum P_1Q_1}{\sum P_0Q_1} \times \frac{\sum Q_1P_1}{\sum Q_0P_1} = \frac{1790}{1910} \times \frac{1790}{1800} \neq \frac{1790}{1740}$$

It does not satisfies facto reversal test.

3. FISHER'S IDEAL INDEX METHOD:

$$\sqrt{\frac{\sum P_1Q_0}{\sum P_0Q_0} \times \frac{\sum P_1Q_1}{\sum P_0Q_1}} \times 100$$

$$= \sqrt{\frac{1800}{1740} \times \frac{1790}{1910}} \times 100 = \sqrt{.96948} \times 100 = 98.462$$

Time Reversal Test

$$= \sqrt{\frac{\sum P_1Q_0}{\sum P_0Q_0} \times \frac{\sum P_1Q_1}{\sum P_0Q_1}} \times \sqrt{\frac{\sum P_0Q_1}{\sum P_1Q_1} \times \frac{\sum P_0Q_0}{\sum P_1Q_0}} \neq 1$$

$$= \sqrt{\frac{1800}{1740} \times \frac{1790}{1910}} \times \sqrt{\frac{1910}{1790} \times \frac{1740}{1800}} = 1$$

It satisfies time reversal test.

Factor Reversal Test

$$\sqrt{\frac{\sum P_1Q_0}{\sum P_0Q_0} \times \frac{\sum P_1Q_1}{\sum P_0Q_1}} \times \sqrt{\frac{\sum Q_1P_0}{\sum Q_0P_0} \times \frac{\sum Q_1P_1}{\sum Q_0P_1}} = \frac{\sum P_1Q_1}{\sum P_0Q_0}$$

$$\sqrt{\frac{1800}{1740}\times\frac{1790}{1910}}\times\sqrt{\frac{1910}{1740}\times\frac{1790}{1800}}=\frac{1790}{1740}$$

It satisfies facto reversal test.

11.15.3 CIRCULAR TEST

Circular test was given by Wester Guard. This test is like Time Reversal test but applied to more number of years. According to this test if data of the different periods is compared by shifting the base, we should be able to get the index of any period by correlating the different base periods used. Symbolically

$$=\frac{P_1}{P_0}\times\frac{P_2}{P_1}\times\frac{P_3}{P_2}=1$$

Only Simple Aggregative, Simple Geometric Mean of price relatives and Kelly's index meet this criteria.

11.16 COST OF LIVING INDEX NUMBER OR CONSUMER PRICE INDEX NUMBER

The price index discussed so far now are general wholesale price index that shows the change in general price level in the economy. These index numbers does not show that how these changes in prices are affecting the life and expenditure of a common consumer. These index numbers does not reflect that how cost of living of the consumer is changing over a period of time. For this purpose we have different index numbers known as Cost of Living Index Numbers or Consumer Price Index Numbers. These index numbers are not based on wholesale prices rather these are based on retail prices. Consumers belonging to different sections have different pattern of consumption, for example rich people have different pattern of consumption than poor people. Similarly, people of different region have different consumption pattern, for example consumption pattern of person living in Delhi is different from consumption pattern of person living Tamil Nadu. So, normally different consumer price index are prepared for different section of people or the people living in different regions of the country. We can define cost of living index as:

It is an index that shows the effect of changes in prices of various goods and services on purchasing power of particular set of persons during a particular period.

It is a weighted index number and generally the expenditure done on particular set of commodities is taken as weight while preparing index number.

11.16.1 Uses of Consumer Price Index Numbers

1. These index numbers measure the change in cost of living so these are helpful in fixation of wages and salary.
2. The index numbers are very useful for government in formulation of the policy specially related to poor people.
3. As there is inverse relation between inflation and value of money, these index numbers are helpful in determining the value of money.
4. These index number help the economic analyst in analysis of various markets.
5. These Index help us to determine the effect of change in prices on the living standard of the people in different geographical regions.

11.16.2 CONSTRUCTION OF CONSUMER PRICE INDEX

1. **SCOPE AND COVERAGE:** Before preparation of these index numbers, we must decide scope and coverage of these index numbers. We have to decide the region to be covered and the section of people to be covered.
2. **ENQUIRY OF FAMILY BUDGET**: Next step is to find family budget that means, expenditure incurred by different families on different commodities. For this purpose adequate number of families must be included.
3. **SELECTION OF BASE YEAR**: This is important task. Base year is the year with which we would compare our current prices. Only in normal year should be taken as base year.
4. **OBTAINING PRICE QUOTATIONS**: This is very important but difficult task. For preparing index number we have to collect data of the prices. As prices of different commodities are different at various places, it is difficult to get this data.
5. **SELECTION OF SUITABLE WEIGHTS:** Next step is to decide the weights that are to be assigned while constructing the index number. Different weights can be assigned, for example we may consider quantity consumed as weight or we may take expenditure incurred as the weight.
6. **SELECTION OF METHOD**: Last step in if cost of living index is to select the method. Normally two methods are applied for this purpose that are Aggregate Expenditure Method and Family Budget Method.

11.16.3 AGGREGATE EXPENDITURE METHOD

This method is similar to Lespeyre's method. In this, quantity of base year is taken as weight. Following are steps of this method:

1. Take the prices of base year and multiply it with quantities of base year.
2. Add up the amount calculated in step 1 to find aggregate expenditure of base year.
3. Multiply the prices of current year with quantities of base year.
4. Add up the amount to get expenditure of current year.
5. Divide the expenditure of current year with the expenditure of base year and multiply the result with 100 to find the index number. Following is the formula.

$$\text{Consumer Price Index (Agrregate Expenditure Method)} = \frac{\sum P_1Q_0}{\sum \ _0Q_0} \times 100$$

11.16.4 FAMILY BUDGET METHOD:

In this method budget of a family is studied for finding the index number. Following are the steps of this method:

1. Calculate price relative by dividing current prices with the base year $\frac{P1}{P_0}$. and such relatives are denoted by R.
2. Calculate expenditure of base year (P_0Q_0), this is done by multiplying prices of base (P_0) year with quantities of base year (Q_0).
3. This expenditure is taken as weight and is represented by W.
4. Find the total expenditure that is $\sum$W.
5. Multiply the relative with the weights and find the product (RW).
6. Calculate the sum of the product calculated above ($\sum$RW).
7. Divide the sum so obtained with the total of weights to obtain index number. Following is the formula.

$$\text{C nsumer Price Index (Family Budget Method)} = \frac{\sum RW}{\sum W} \times 100$$

Example 7: From the following data find out consumer price index number for the year 2020 taking 2018 as base by using (i) the aggregate expenditure method, and (ii) the family budget Method

Commodities	Quantity 2018	Price in 2018 (Rs.)	Price in 2020 (Rs.)
A	**6 Kg**	**11.50**	**12.00**
B	**6 Kg**	**10.00**	**16.00**
C	**1 Lt**	**12.00**	**18.00**
D	**6 Kg**	**16.00**	**20.00**
E	**4 Kg**	**4.00**	**3.00**
F	**1 Kg**	**40.00**	**30.00**

Solution:

Commodities	Quantities 2018	Prices in 2018	Prices in 2020	P_1Q_0	P_0Q_0

	Q_0	P_0	P_1		
A	6	11.50	12.00	72	69
B	6	10.00	16. 00	96	60
C	1	12.00	18.00	18	12
D	6	16.00	20.00	120	96
E	4	4.00	3.00	12	16
F	1	40.00	30.00	30	40
				$\sum P_1Q_0 = 348$	$\sum P0_1Q_0 = 293$

$$\text{C. P. I (Agrregate Expenditure Method)} = \frac{\sum P_1Q_0}{\sum P_0Q_0} \times 100 = \frac{348}{293} \times 100 = 118.77$$

Commodities	Quantity Consumed Q_0	Price in 2018	Price in 1983	Pi -----x 100 Po	P_0Q_0 W	RW
		P_0	P_1	R		
A	6	11.50	12.00	104.35	69	7200.15
B	6	10.00	16.00	160.00	60	9600.00
C	1	12.00	18.00	150.00	12	1800.00
D	6	16.00	20.00	125.00	96	12000.00
E	4	4.00	3.00	75.00	16	1200.00
F	1	40.00	30.00	75.00	40	3000.00
					$\sum W = 293$	$\sum RW = 34800.15$

$$\text{C. P. I (Family Budget Method)} = \frac{\sum RW}{\sum W} \times 100 = \frac{34800.15}{293} \times 100 = 118.77$$

11.17 TEST YOUR UNDERSTANDING (B)

1. Find Laspeyre, Paasche and Fisher Index from following

Item	Price of the Base Year	Qty of the Base Year	Price of the Current Year	Qty of the Current Year
A	12	20	15	25

B	10	8	16	10
C	15	2	12	1
D	60	1	65	1
E	3	2	10	1

2. Calculate Laspeyre, Paasche, Bowley, Fisher, Marshal and Edgeworth price Index from following

Item	Qty of of the Base Year	Expenditure of the Base Year	Qty of the Current Year	Expenditure of the Current Year
A	10	120	12	156
B	50	700	40	600
C	15	240	25	475
D	12	216	15	240

3. Calculate Laspeyre, Paasche, Fisher, Marshal and Edgeworth price Index from following

Item	Price 2015	Qty 2015	Price 2017	Qty 2017
A	5	100	6	150
B	4	80	5	100
C	2.5	60	5	72
D	12	30	9	33

4. Calculate index by using Weighted price relative method

Item	Price 2015	Price 2017	W
A	10	12	10
B	15	19	15
C	20	25	8
D	25	28	12

5. Apply Laspeyre, Paasche and Fisher Method on the following data and check whether these methods satisfy Time Reversal and Factor Reversal Test or not

Item	P_0	Q_0	P_1	Q_1
A	5	15	5	5
B	7	5	4	3
C	8	6	6	10
D	3	8	3	4

6. From the following data find out consumer price index number for the year 2020 taking 2018 as base by using (i) the aggregate expenditure method, and (ii) the family budget Method

Commodities	Quantity 2018	Price in 2018 (Rs.)	Price in 2020 (Rs.)
A	100	8	12
B	25	6	7.5
C	10	5	5.25
D	20	48	52
E	25	15	16.5
F	30	9	27

Answers:

1. Laspeyre - 129.09, Paasche – 130.13, Fisher 129.61
2. L = 106.35, P = 107.06, B = 106.72, F = 106.75, M & E = 106.7
3. L = 118.05, P = 119.18, F = 118.61, M & E = 118.68
4. 120.97
5. L= 85.165, P = 86.232, F = 85.697, only Fisher method satisfy both test. 6. 142.13

8.14 LET US SUM UP

- **Index number shows change in variable over a period of time.**
- **Price index shows change in price in current year in comparison to base year.**
- **Normally the base of index is taken as 100.**
- **There are different types of index like price index, quantity index, value index.**
- **Index number can be prepare without assigning weights or after assigning weights.**
- **Popular weighted aggregative index are Laspeyre, Paasche, Bowley, Fisher, Marshal Edgeworth and Kelly.**
- **There are test to check consistency of the index number.**
- **Only Fisher index satisfy Time Reversal and Factor Reversal tests.**
- **Consumer price index shows change in cost of living of the consumer.**

8.15 KEY TERMS

- **INDEX NUMBERS:**An index number is a device with help of which we can measure the relative change in one variable over a period of time. Normally while preparing the index number, we compare the current year variable with the variable of as base year. The index number of the base year is mostly taken as 100
- **PRICE INDEX NUMBERS**: These index numbers are used for measuring the change in prices of the commodities over a period of time. In other words we can say that these index numbers find the change in value of money over a period of time. These index numbers are most popular index number. These Index numbers may be based on Wholesale Price Index or Retail Price Index.
- **QUANTITY INDEX NUMBERS**: The Quantity or Volume Index Numbers measure the change in quantities used by people over a period of time. Under these index numbers, we calculate change in physical quantity of goods produced, consumed or sold over a period of time. There are different types of quantity index numbers such as Agricultural Production Index Number, Industrial Production Index Number, Export Import Index Number etc.
- **VALUE INDEX NUMBERS**: Value Index Numbers compare the change in total value over period of time. These index numbers takes into consideration both prices and quantity of the product while finding the change over a period of time. These Index Numbers are very useful in finding consumption habits of the consumers.
- **TIME REVERSAL TEST:** This test was suggested by Fisher. According to this test, an ideal index number is one which works both ways that i.e. backward and forward. In other words we can say that following condition should be satisfied: $\mathbf{P_{01} \times P_{10} = 1}$
- **FACTOR REVERSAL TEST (F.R.T.):** This test was also suggested by Fisher. According to this test, an ideal index number is one which does not give inconsistent result if we change price with quantity and quantity with price. According to this test when we multiply change in price with change in quantity the ratio must be equal to total change in value. $\mathbf{P_{01} \times Q_{01}} = \frac{\sum P_1 Q_1}{\sum P_0 Q_0}$.

8.16 REVIEW QUESTIONS

1. What are index numbers? What are its uses?
2. Explain problems faced in construction of index numbers.
3. What are different types of Index numbers.
4. Explain different steps in construction of index numbers.
5. What are different methods of construction of index numbers?
6. Explain Simple Aggregative Index numbers. What are its Merits and Limitations?
7. What are Simple Price Relative Index numbers? What are its Merits and Limitations?
8. Explain Weighted Aggregative Index numbers. What are its Merits and Limitations?
9. What are Weighted Price Relative Index numbers? What are its Merits and Limitations?
10. What are tests of consistency of index numbers. Give various tests of consistency.

11. Explain Time Reversal and Factor Reversal Test.
12. Why Fisher's Index is known as Ideal Index Number.
13. What are cost of living index. Give its methods of construction. 14. What are consumer price index numbers? What are its uses.

8.19 FURTHER READINGS

1. J. K. Sharma, *Business Statistics*, Pearson Education.
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4. Richard Levin and David S. Rubin, *Statistics for Management*, Prentice Hall of India, New Delhi.

B. COM (DIGITAL)

SEMESTER II

COURSE: BUSINESS MATHEMATICS AND STATISTICS

UNIT 12 – TIME SERIES ANALYSIS

STRUCTURE

12.0 OBJECTIVES

After studying the Unit, students will be able to

- Define the Meaning of Time series Analysis.
- Distinguish different types fluctuations in the time series analysis.
- Understand how time series analysis is useful for forecasting.
- Apply various methods of time series in prediction of trends.

12.1 INTRODUCTION

One of the important functions of business managers is to make forecast about the future. This forecasting helps them in making the business decisions. There are many Statistical Techniques that helps a business manager in forecasting of the future. Time series analysis is one such technique. This technique is not only used by Business managers, rather other persons interested in forecasting also use this technique like economists etc. Time series analysis is a tool with help of which we try to predict the future values on the basis of data available with us. For example if we have data of sales of a company for last 10-12 years and we want to predict the likely sale of the company for the next year, we can do so using time series analysis. Following are few examples of time series analysis:

- A series of data related to production of goods, prices of goods or consumption level of goods.
- Data related to the rainfall or temperature of a region.
- The data related to sales profit etc of any business firm.
- The data related to exports and imports of the country.
- The data related to population, birth rate or death rate in a country.

12.2 DEFINITION OF TIME SERIES

In time series we collect the data related to statistical observations and place such data in chronological order, that means in the order of occurrence of these observations. On the basis of these observations we can try to predict the future values of the observation. Following is the definition of time series analysis:

According to Ya-Lun-Chou "A time series may be defined as a collection of readings belonging to different time periods of some economic variable or composite of variables".

According to W.Z. Hirsch "The main objective in analyzing time series is to understand, interpret and evaluate change in economic phenomena in the hope of more correctly anticipating the course of future events".

12.3 ESSENTIAL CONDITIONS OF TIME SERIES ANALYSIS

1. Time series analysis must consist of those values that are homogenous in nature for example the sales data of every year must be in same quantities like in kilograms. If sales of some years are given in quantity and other are given in value, then we cannot apply time series analysis.
2. The data present must be in reference to time only. So, out of two variables given, one variable should be time. For example if relation between Price and Demand is given it is not time series.
3. The data must be arranged in chronological order.

4. The data must be available for long period of time at least 10 to 12 years.
5. We must try to keep the equal gap between two periods.
6. If the gap between the periods is not equal and some values are missing, we should try to find out those values using the interpolation.
7. The data must have some relation with the time. For example if we are measuring average marks of the students nn a class, it is not related to time.

12.4 ADVANTAGES OF TIME SERIES ANALYSIS

1. Time series analysis helps us in understanding the past behaviour of the phenomenon.
2. It help us in predicting the future course of action.
3. It helps us in understanding that how values are changing with the passage of time.
4. With help of time series, we can isolate impact of various factors like seasonal factors, cyclical factors or other irregular changes in the data.
5. With help of time series, we can find deviations between the actual achievements and the expected achievements.

12.5 COMPONENTS OF TIME SERIES

A large number of forces are there that affect the data. For example if the sales of a company are changing with the passage of time, there are many forces responsible for it. We can classify these forces basically into four categories known as components or elements of the time series. Following are these components:

1. Secular Trend
2. Seasonal variations
3. Cyclical variations
4. Irregular variations

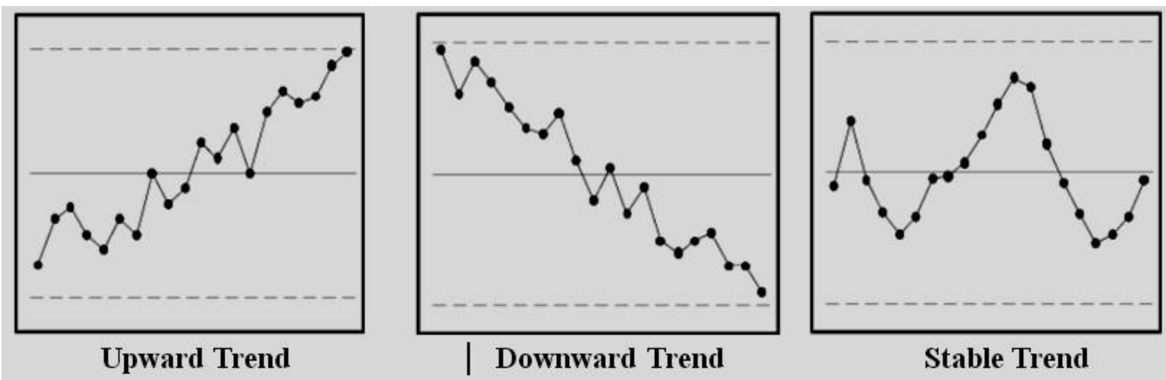
12.6 SECULAR TREND

The word secular is taken from latin word 'Saeculum', which means a 'Generation'. So, as the names suggests, secular Trends are long-term Trends which normally occurs over it period of 15 to 20 years. Sometime, these trends may show upward results and other time it may show downward results. For example we can see that number of persons who are travelling by air is increasing over a period of time. Similarly, we can see that infant mortality rate in the country is decreasing over a period of time. These both are secular trend but one trend is showing upward result and other trend is showing downward result.

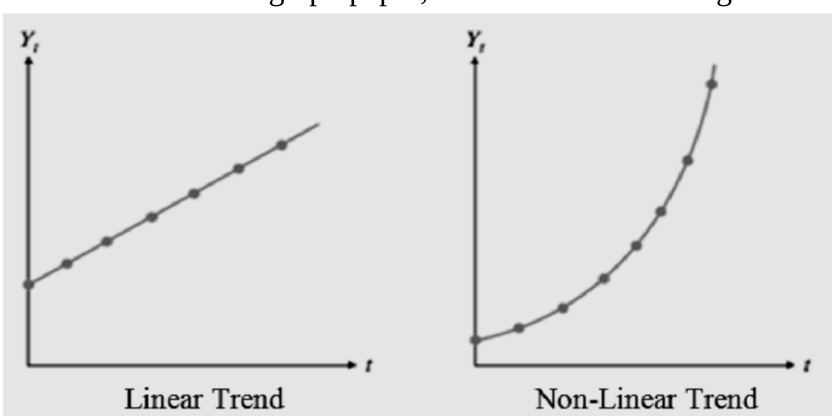
12.6.1 FEATURES OF SECULAR TREND

Following are the characteristics of secular trends:

1. These Trends are related to long period of time.
2. These trends result of factors that are more or less stable in nature. For example the taste of people change of Technology takes time and does not happen overnight. 3. These Trends may show upward, downward or stable results.



4. These trends may be linear or nonlinear in nature. Linear Trends are those Trends which change proportionately with the passage of time and these are presented in a graph as a straight line. Non linear friends are those which does not change proportionately, so when we draw these trends on a graph paper, these are not in a straight line.



12.6.2 USES OF SECULAR TREND

Following are the benefits of studying secular trends:

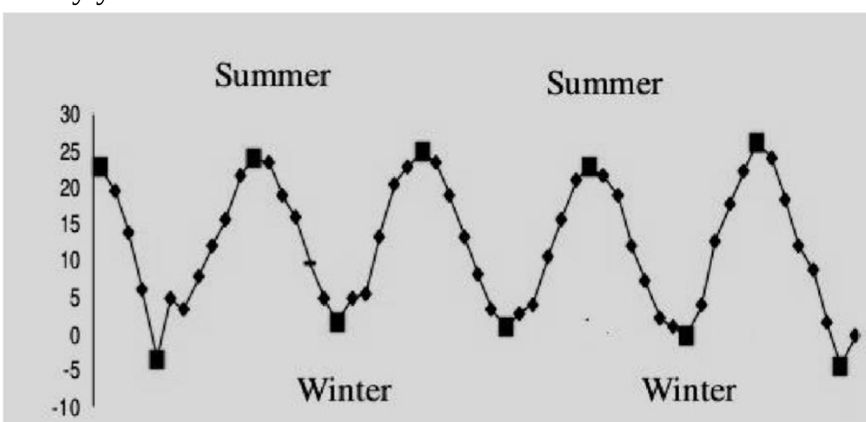
1. As these trends are long-term, these help us in understanding direction of change. We can find whether phenomenon is increasing, decreasing or is stable with the passage of time.
2. These Trends can help us in predicting the future.
3. Secular Trends can help us in comparing two or more series and we can see which series is changing more rapidly.

4. It can be used extrapolation of the future values. 5. With the help of this trend, we can also study impact of other components of time series.

12.7 SEASONAL VARIATIONS:

Seasonal variations are short term variations. These variations occur regularly and their trend is repetitive. These variations may occur on yearly basis, half yearly basis, monthly basis, weekly basis or any other time period basis. There may be many reasons of these variations but these variations generally occur due to following two reasons:

1. **CLIMATIC CONDITIONS**: Sometime seasonal variations take place due to climate change. We can see that there are climatic cycle occur during the year. This climatic cycle also effect on many things like sales of company, consumption pattern etc. For example in rainy season sale of umbrellas increase, in summer season sale of air conditioners increase and similarly during the winter season sale of woolen clothes increase. These variations take place every year.
2. **CUSTOMS AND TRADITIONS**: Sometime seasonal variations take place due to customs and traditions. For example in India is tradition of purchasing new items in the household at the time of Diwali festival. So this is also seasonal variation which take place every year.



12.7.1 FEATURES OF SEASONAL VARIATIONS

Following are characteristics of seasonal variations:

1. These variations are short duration variations.
2. These variations repeat periodically.
3. It may have both upward and downward trend, for example in winter sale of woolen garments increase but at the same time sale of soft drinks decrease.
4. These variations may occur on yearly, quarterly monthly or weekly basis.
5. As these variations are repetitive and short duration in nature, these are comparatively easy to analyse.

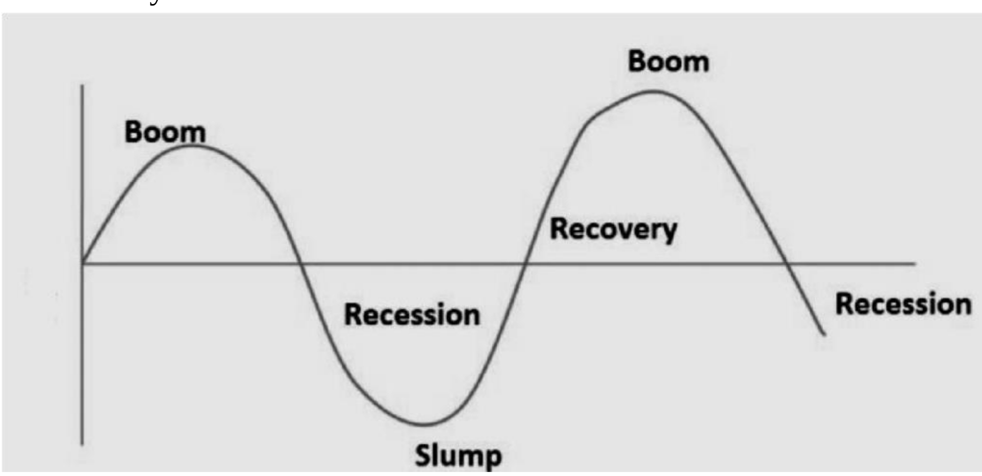
12.7.2 USES OF SEASONAL VARIATIONS

Following are the uses of seasonal variations

1. Analysis of these variations is very important for business in planning their production schedule. Business can decide its production according to the seasonal variations.
2. Seasonal variations are useful for consumers also as they can plan their purchases according to the season. They know in advance which season is coming, so they can purchase items related to that season.
3. Seasonal variations also help consumers in making a bargain. Consumers may get off season items at lower price.
4. These variations help us in separating the effect of cyclical and irregular variations.
5. Business can use seasonal variations in many decisions like purchasing, inventory control, recruitment of employees, advertising etc. 6. These variations help us in making short term forecast.

12.8 CYCLICAL VARIATIONS

As the term 'cycle' suggest, these variations are recurrent variations. These variations are long Run variations and shows the recurring pattern of rise and decline. These variations are also known as oscillating movements. These variations do not have any fix duration. Sometime one cycle may be complete in 2-3 years, but some other time it may takes 7-8 years to complete. For example a business cycle is cyclical variation that has four phases Boom, recession, depression and recovery.



12.8.1 FEATURES OF CYCLICAL VARIATIONS

Following are features of cyclical variations:

1. Normally these occur over a long period that is more than 1 year.
2. There is no fixed duration of these variations, time one cycle is completed in 3 years but some other time it may take 7 years.

3. These variations are oscillating in pattern. 4. These are comparatively more difficult to measure.

12.8.2 USES OF CYCLICAL VARIATIONS

Following are uses of cyclical variations

1. Cyclical variations can help business in Planning its strategy. Business can plan strategy according to Boom or depression in the market.
2. Analysis of these variations can help business in predicting the turning points of cyclic variations.
3. Analysis of these variations can help business planning stabilization policies like diversification etc.
4. Analysis of these variations can help us in finding irregular variations.
5.

12.9 IRREGULAR VARIATIONS

From the name of these variations, it is clear that these variations do not have any definite pattern and are irregular in nature. These variations do not have any fixed time period and occur due to accidental or random factors like strike, floods, pandemic, war, earthquakes etc.

12.9.1 FEATURES OF IRREGULAR VARIATIONS

Following are features of irregular variations.

1. These variations do not have any fixed pattern.
2. Mostly these variations are short duration.
3. These variations are very difficult to predict. 4. These occur due to random or accidental such as floods quake war etc.

12.10 DE-COMPOSITION OF TREND

As we have discussed above that any time series data comprise of various components namely Secular trend, seasonal variations, cyclical variations or irregular variations. In the time series Analysis we try to identify various components of time series separately. This can be done by measuring the impact of one component while we keep other component constant. This process of finding each of the element of time series separately is known as De-composition of time series. There are many models which are normally used to analyze the time series. These are:

12.10.1 ADDITIVE MODEL:

This model of decomposition assumes that the four elements of time series are not dependent on each other and does not affect each other. Each trend operate independently. So if we have to measure overall trend of the time series, it is combination of all the four elements. By adding effect of all the elements we can get the overall time series trend. Mathematically we can say that

Y=T+S+C+I

Shortterm fluctuations= Y – T= S+C+I

Cyclical and Irregular Fluctuation = Y – T – S = C+I

Irregular Fluctuation = Y – T – S– C = I

where

Y = time series value,

T = Secular Trend Variations,

S = Seasonal Variations,

C = Cyclical Variations and

I = Irregular Variations.

Though in additive model we assume that all the elements operates independently, but in reality it is not true as all the elements have significant effect on each other and this is the major limitation of additive model.

12.10.2 MULTIPLICATIVE MODEL:

The Multiplicative model are based on the assumption that all the components of the time series are related to each other and have significant effect on each other. So, if we want to calculate overall trend, it cannot be calculated by simply adding the four components. Rather it is multiple effect of all the four elements. So according to this model overall trend is

Y=T × S × C × I

Short term fluctuations = $\frac{Y}{T}$ **= S × C × I**

Cyclical and Irregular Fluctuation = $\frac{Y}{T \times S}$ **= C × I**

Irregular Fluctuation = = $\frac{Y}{T \times S \times I}$ **= I**

Here it is important to mention that the values of S, C and I are not absolute values rather these are relative variations and these are expressed in relative change or some indices.

12.10.3 DIFFERENCE BETWEEN ADDITIVE MODEL AND MULTIPLICATIVE MODEL

Sr. No.	Additive Model	Multiplicative Model

1	It is based on the assumption that all elements of time series are independent of each other.	It is based on the assumption that all elements of time series are dependent o each other.
2	Under this model the overall trend can be found by adding the four elements.	Under this model the overall trend can be found by multiplying the four elements.
3	Under this model absolute values of the four elements are taken for calculating overall trend.	Under this model relative values of the four elements are taken for calculating overall trend.

12.11 MEASUREMENT OF TREND

Trend means the direction or tendency of series of data over a long period of time. We want to know that whether the values are increasing over a period of time, decreasing over a period of time or these are stable over a period of time. This is known as Trend. We generally assume that the past behaviour of the data will continue in future as well, so finding the trend could help us in predicting the future. There are four methods of finding the trend which are as follows:

- Free hand graphic method
- Semi average method
- Moving average method
- Method of least square

12.12 FREE HAND GRAPHIC METHOD

This is the simplest method of finding the trend and is very flexible in nature. This method is also known as 'free hand curve fitting method'. Following are the steps of finding trend under this method:

1. In the graph paper line chart is to be drawn.
2. For this purpose time is taken on x-axis whereas values are taken on y axis.
3. Plot all the given values in the graph paper.
4. Then we join all the points in the graph paper to show the actual value.
5. After that smooth straight line is drawn which pass through middle of the actual values drawn. 6. This line is the trend line.

12.12.1 POINTS TO BE CONSIDERED IN DRAWING FREE HAND GRAPHIC METHOD

Following are precautions that be taken while drawing trend line

1. We must try to draw smooth line.

2. We must try that number of points above the line and the number of points below the line should be equal.
3. If there are cycles in the data, number of cycles above the line and number of cycles below the line should be equal.
4. We must try that trend line should pass through middle of the points.
5. We should try to keep sum of vertical distance between trend line and the points nearly zero, that means we must try to have minimum deviation.

12.12.2 MERITS OF FREE HAND GRAPHIC METHOD

Following are the benefits of graphic method

1. It is simple to draw
2. It does not need any calculations
3. This is very flexible method and is not affected by the fact that data is linear on or non linear. 4. An experienced statistician can use this tool very effectively.

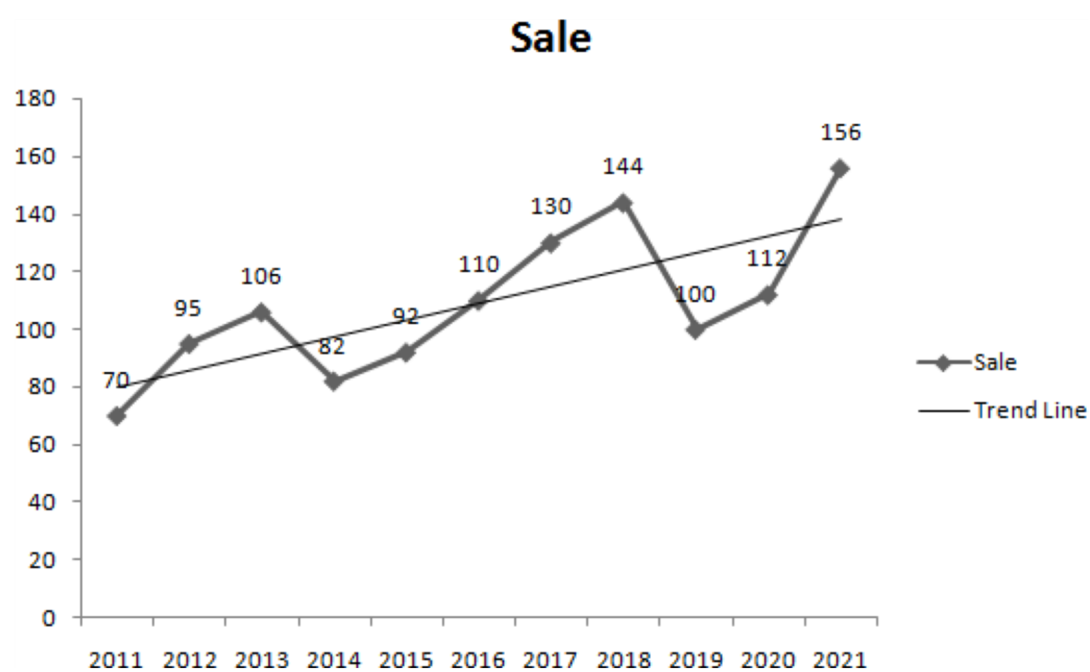
12.12.3 LIMITATIONS OF FREE HAND GRAPHIC METHOD

1. This is very crude method.
2. This method is very subjective and there are chances of personal biasness.
3. It needs lot of experience to draw this chart.
4. With the change in scale of graph there is change in trend also.

Example 1: Fit the straight line graphic curve from the following data:

Year	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**	**2021**
Sale	**70**	**95**	**106**	**82**	**92**	**110**	**130**	**144**	**100**	**112**	**156**

Solution:



From the above graph we can predict any value with the help of trend line.

12.13 TEST YOUR UNDERSTANDING (A)

1. Following is the data of Harshit Ltd. draw a straight trend line using free hand graphic method.

Year:	2009	2010	2011	2012	2013	2014	2015	2016	2017
Sales (in '000 kg):	20	22	24	21	23	25	23	26	24

2. On basis of following data fit straight trend line using free hand graphic method.

Year	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021
Production	64	82	97	71	78	112	115	131	88	100	146

12.14 SEMI AVERAGE METHOD

Semi average method is second method of finding the trend line. This is an objective method and is not merely based on guesswork. Under this method it is very easy to find trend line. Following are the steps of semi average method:

1. Divide the series in two equal parts, for example if there are 10 values take 5 values in each part.
2. In case of number of values are in odd number, middle value may be left and remaining values can be divided into two parts. For example if there are 11 values, 6th value may be left and will have two parts having five values each.
3. Find the Arithmetic mean of both the parts.
4. These arithmetic means are called semi averages.
5. Now these semi averages are plotted in the graph as points against middle of each time period for which these have been calculated.
6. Join the points to find out straight line Trend.

12.14.1 MERITS OF SEMI AVERAGE METHOD

1. This is simple and easy to draw.
2. This is objective method and does not suffer from limitation of biasness.
3. As the line drawn is extendable on both sides we can predict future values also.

12.14.2 LIMITATIONS OF SEMI AVERAGE METHOD

1. This method is useful only for Linear trends. 2. This method is based on Arithmetic mean which is not a perfect average.

Even Number of Years

Example 3 : From the data given below find semi average trend line and also find out trend values.

Year	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**
Sale ‘000’	**80**	**61**	**76**	**73**	**62**	**50**	**45**	**65**	**55**	**35**

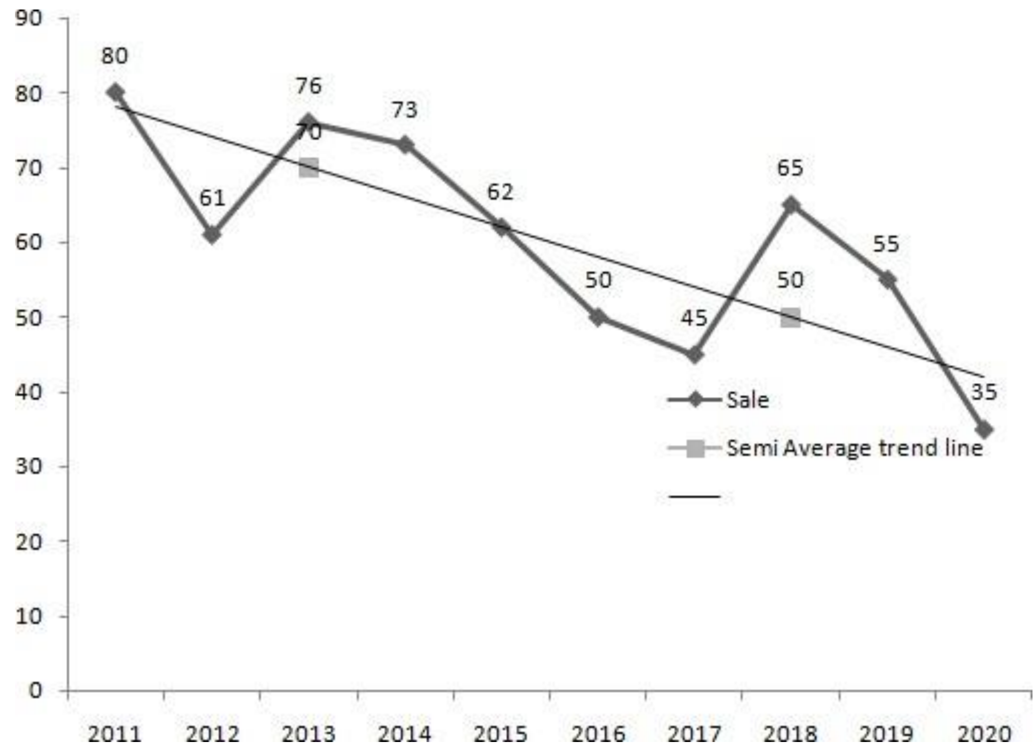
Solution

As the number of years is even, we have got two blocks of five years each. Now we will find arithmetic mean of these two blocks and will write against middle of the block.

2017	45	
2018	65	$= \frac{50+45+65+55+35}{5} = \frac{250}{5} = 50$
2019	55	
2020	35	

For finding the trend in the graph 70 is plotted against year 2013 and 50 is plotted against the year 2018.

Year	Sale ‘000’	Semi Average
2011	80	
2012	61	
2013	76	$= \frac{80+61+76+73+62}{5} = \frac{350}{5} = 70$
2014	73	
2015	62	
2016	50	



Annual increment = $\frac{Difference\ in\ Semi\ Average\ values}{Difference\ in\ two\ years\ to\ which\ Semi\ Average\ belongs}$

Annual increment = $\frac{50-70}{2018-2013} = \frac{-20}{5} = -4$

As we can see from the above data that semi average is showing downward trend so this annual increment will be deducted to semi average of 2013 onwards. For finding the values of the years before 2013 it will be added to the value every year. So trend values are:

Year	Actual Sale '000'	Trend Sale '000'
2011	80	78 (74+4)
2012	61	74 (70+4)
2013	76	70
2014	73	66 (70-4)
2015	62	62 (66-4)
2016	50	58 (62-4)

2017	45	54 (58-4)
2018	65	50 (54-4)
2019	55	46 (50-4)
2020	35	42 (46-4)

Odd Number of Years

Example 4: From the data given below find semi average trend line and also find out trend values.

Year	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**	**2021**
Sale '000'	**70**	**96**	**108**	**82**	**94**	**110**	**128**	**142**	**98**	**112**	**150**

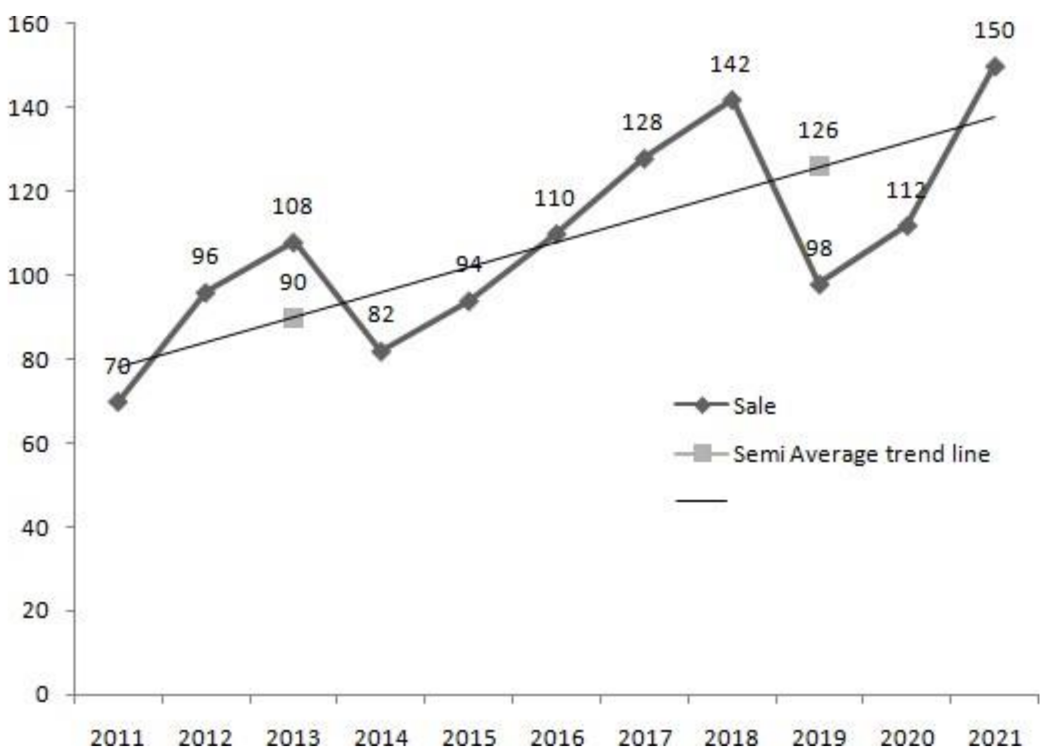
Solution:

As the number of years is odd, the middle year 2016 is left and we have got two blocks of five years each. Now we will find arithmetic mean of these two blocks and will write against middle of the block.

Year	Sale '000'	Semi Average
2011	70	
2012	96	
2013	108	$= \frac{70+96+108+82+94}{5} = \frac{450}{5} = 90$

2014	82	
2015	94	
2016	110	
2017	128	
2018	142	
2019	98	$= \frac{128+142+98+112+150}{5} = \frac{630}{5} = 126$
2020	112	
2021	150	

For finding the trend in the graph 90 is plotted against year 2013 and 126 is plotted against the year 2019.



Annual increment = $\frac{Difference\ in\ Semi\ Average\ values}{Difference\ in\ two\ years\ to\ which\ Semi\ Average\ belongs}$

Annual increment = $\frac{126-90}{2019-2013} = \frac{36}{6} = 6$

As we can see from the above data that semi average is showing upward trend so this annual increment will be added to semi average of 2013 onwards. For finding the values of the years before 2013 it will be deducted from value every year. So trend values are:

Year	Actual Sale '000'	Trend Sale '000'
2011	70	78 (84-6)
2012	96	84 (90-6)
2013	108	90
2014	82	96 (90+6)
2015	94	102 (96+6)
2016	110	108 (102+6)
2017	128	114 (108+6)
2018	142	120 (114+6)
2019	98	126 (120+6)
2020	112	132 (126+6)
2021	150	138 (132+6)

12.15 TEST YOUR UNDERSTANDING (B)

1. FROM THE PRODUCTION OF MAHANTA LTD FIT A STRAIGHT LINE TREND USING SEMI AVERAGE METOD:

Year	2011	2012	2013	2014	2015	2016	2017	2018
Prdocution ('000 Units)	200	210	218	192	204	216	224	228

Also predict value of 2020.

2. FIT STRAIGHT LINE TREND USING SEMI AVERAGE METHOD

Year	2012	2013	2014	2015	2016	2017	2018

Sales (in thousand units)	101	106	114	110	109	115	112

3. **SALES OF ABHINAV ARE GIVEN, FIT A STRAIGHT LINE TREND USING SEMI AVERAGE METOD:**

Year	2012	2013	2014	2015	2016	2017	2018
Sales (‘000 Units)	80	90	92	83	94	99	92

Also predict value of 2021.

Answers

1. 230

3. 84

12.16 MOVING AVERAGE METHOD

Under this method we try to find out trend line using the concept of moving average. For this, first of all we decide the period for which moving average is to be calculated, for example we can take 3 year moving average, 4 years moving average, 5 year moving average or so on.

Following are the steps in this method

1. First of all decide the length of period for which moving average will be taken.
2. Calculate the moving average of first group starting with first item.
3. After that find out moving average of second group leaving the first item.
4. Repeat this process until moving average is calculated for all the groups ending with last item.
5. Write the first moving average in front of the middle item of the group.
6. Repeat this process till all the moving averages are placed front of middle item of the group.
7. In case, even number of years are taken as period of moving average, the moving average is placed in middle of the period and then average of the adjacent averages are placed against mid item.

12.16.1 ADVANTAGES OF MOVING AVERAGE

1. This method is easy to adopt.
2. It is flexible method and any period can be taken as moving average upon the period of cyclical Trend.
3. This method is free from bias.
4. Moving average reduce the impact of cyclical variations.

5. This method is not only useful for measurement of trend but could also help in finding seasonal, cyclical and irregular variations.

12.16.2 LIMITATIONS OF MOVING AVERAGE METHOD

1. This method cannot be used for predicting the future values.
2. We cannot calculate trend for all the years as items beginning and some items at the end are lost.
3. It is very difficult to decide the period of moving average.
4. This Method is greatly affected by presence of extreme values.
5. This method is not useful we are estimating non linear Trend.

Odd period Moving Average

Example 5: Calculate 3 yearly and 5 yearly moving averages for the following data:

Year	**2009**	**2010**	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**
Sales	**52**	**49**	**55**	**49**	**52**	**57**	**54**	**58**	**59**	**60**	**52**	**48**

Solution:

Following are the steps for 3 yearly moving average

1. Fist, compute the total of value of first three years (2009, 2010, 2011) and place the three year total against the middle year 2010.
2. Now, leaving the first year's value, add up the values of the next three years (2010, 2011, 2012) and place the three year total against the middle year 2011.
3. Repeat the process till last year's value i.e. 2020 is taken up.
4. Now divide the three year's total by 3 to get the average and place it in the next column. All these value represent the required trend values for the given year.
5. Same process can be repeated for 5 yearly moving average.

Year	Sale	3 Year Moving Total	3 Year Moving Average	5 Year Moving Total	5 Year Moving Average
2009	52				
2010	49	156	52		
2011	55	153	51	257	51.4
2012	49	156	52	262	52.4
2013	52	158	52.7	267	53.4
2014	57	163	54.1	270	54
2015	54	169	56.3	280	56
2016	58	171	57	288	57.6
2017	59	177	59	283	56.6
2018	60	171	57	277	55.4

2019	52	160	53.3		
2020	48				

Even period Moving Average:

Example 6 : Calculate 3 yearly and 5 yearly moving averages for the following data:

Year	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**
Sales	**250**	**260**	**275**	**300**	**290**	**310**	**318**	**325**	**350**	**340**

Solution:

Following are the steps for 4 yearly moving average

1. Fist, compute the total of value of first four years (2011, 2012, 2013, 2014) and place the fouryear total in between 2nd and 3rd year i.e. between 2012 and 2013..
2. Now, leaving the first year's value, add up the values of the next four years (2012, 2013, 2014, 2015) and place the total b 2011 between 2013 and 2014.
3. Repeat the process till last year's value i.e. 2020 is taken up.
4. Now divide the four year's total by 4 to get the average and place it in the next column. All these value represent the required trend values for the given year.
5. Divide the first two four yearly average by 2 to get the required trend values corresponding to the given years as shown in the table:

Year	Value	4 Yearly Total	4 Yearly Average	Trend Value
2011	250			
2012	260			
		1085		
2013	275		271.25	276.25
		1125	281.25	
2014	300			287.5
		1175	293.75	
2015	290			299.12
		1218	304.5	
2016	310			307.63
		1243	310.75	
2017	318			318.25
		1303	325.75	
2018	325			329.5
		1333	333.25	
2019	350			

2020	340

12.17 TEST YOUR UNDERSTANDING (C)

1. Calculate 3 yearly moving averages for the following data:

Year	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Sales	11200	12300	10600	13400	13800	14500	11600	14300	13600	15400

2 Calculate 5 yearly moving averages for the following data:

Year	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
No. of Employees	332	317	357	392	402	405	410	427	405	438

3 Calculate 4 yearly moving averages for the following data:

Year	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Value	100	105	115	90	95	85	80	65	75	70	75	80

Answers

1. **11366.7, 12100, 12600, 13900, 13300, 13466.7, 13166.7, 14433.3**
2. **360, 374.6, 393.2, 407.2, 409.8, 417**
3. **101.875, 88.75, 91.875, 84.375, 78.75, 74.375, 71.875, 73.125**

12.18 LEAST SQUARE METHOD

This is most scientific and popular method of finding the trend line. Under this method the lines of best fit is drawn as the lines trend.. These lines is known as the lines of the best fit because, with help of these lines we can make the estimate of the values of variable according to the different time period. According to the Least Square method, trend line should be plotted in such a way that sum of square of the difference between actual value and estimated value of the dependent variable should be least or minimum possible. Mathematically this line is represented by

$Y_c = a + bX$

Where Y_c – Computed Trend Value

X – Independent Variable represented by time a
& b – Constants

12.18.1 DIRECT METHODS TO ESTIMATE TREND LINE

Following are steps for finding trend line with help of Direct Method:

3. Take the problem with two variable with X variable as time and other variable for which trend is to be computed like sales, population etc represented by Y.
4. Assume first year as base year and put the value '0' against it.
5. Now put value 1 against second year, 2 against third year and so on till all the years are covered.
6. Now find the values of $\sum X$, $\sum X^2$, $\sum XY$ from the given values.
7. Put these values in following equation: $\Sigma Y = na + b\Sigma X$

   $\Sigma XY = a \Sigma X + b\Sigma X^2$
8. Solve these equations simultaneously and find the values of 'a' and 'b'.
9. Put value of 'a' and 'b' in trend equation Yc = a + bX. 10. Now this trend equation can be used for finding the trend values.

Example 7. The data of sales of Alpha Ltd is given for last 9 years. On the basis of the data find trend value of the year 2021 using the method of least square.

Year	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**
Sales '000'	**10**	**12**	**15**	**20**	**30**	**40**	**50**	**60**	**70**

Solution:

Year	X	Sales (Y)	X^2	XY
2012	0	10	0	0
2013	1	12	1	12
2014	2	15	4	30
2015	3	20	9	60
2016	4	30	16	120
2017	5	40	25	200
2018	6	50	36	300
2019	7	60	47	420
2020	8	70	64	560
	$\sum X = 36$	$\sum Y = 307$	$\sum X^2 = 204$	$\sum XY = 1702$

This is given by Y = *a* + *bX* where *a* and *b* are the two constants which are found by solving simultaneously the two normal equations as follows :

$\Sigma Y = na + b\Sigma X$

$\Sigma XY = a \Sigma X + b\Sigma X^2$

Substituting the given values in the above equations we get,

307 = 9a + 36b .... (i)

1702 =36a + 204b .... (ii)

Multiplying the eqn. (i) by 4 we get

1228 = 36a + 144b .... (iii)

Subtracting the equation (iii) from equation (ii) we get,

1702 =36a + 204b

-1228 = -36a - 144b

474 = 60b

or b = 7.9

Putting the above value of b in the eqn. (i) we get,

307 = 9a+ 36(7.9) or 9a

=307- 284.4 or

a = 2.51

Thus, a = 16, and b = 1.6

Putting these values in the equation Y = *a* + bX we get

Y = 2.51+ 7.9X

So if we want to calculate the trend value of the year 2021 the value of X will be 9 (as 2012 is our base year and its value is 0), the value of Y will be

Y = 2.51+ 7.9 (9) = 73.61

12.18.2 SHORT CUT METHOD

In the direct method we take starting year as the base year. But in case we take the middle period as base yare we can save lot of time and calculation because when middle period is take as the base period the value of ∑X will be 0, hence the two simultaneous equations will become very easy in that case.

Equation (i) $\Sigma Y = na + b\Sigma X$

If $\Sigma X = 0$ than $\Sigma Y = na$

$a = \frac{ZY}{n}$

Equation (ii) $\Sigma XY = a \Sigma X + b\Sigma X^2$

If $\Sigma X = 0$ than $\Sigma XY = b\Sigma X^2$

$b=\frac{ZXY}{\Sigma K^2}$

Odd number of Years

Example 8. The data of sales of Mahesh and Co is given for last 7 years. On the basis of the data find trend line using the method of least square and find trend value of 2021.

Year	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**

Sales ‘000’	**672**	**824**	**967**	**1204**	**1464**	**1758**	**2057**

Solution:

Since the number of years is odd, 2017 is taken as base year with value 0 and one year is taken as one unit.

2

Year	X	Sales (Y)	X	XY
2014	-3	672	9	-2016
2015	-2	824	4	-1648
2016	-1	967	1	-967
2017	0	1204	0	0
2018	1	1464	1	1464
2019	2	1758	4	3516
2020	3	2057	9	6171
	∑X = 0	∑Y =8946	$\sum X^2$= 28	∑X Y =6520

As ∑X is 0, we can apply short cut method a

$$= \frac{ZY}{n} = \frac{8946}{7} = 1278$$

$$\frac{ZXY}{\sum K^2} = \frac{6520}{28} = 232.9$$

b=

Putting these values in the equation Y = *a* + bX we get

Y = 1278+ 232.9 X

So if we want to calculate the trend value of the year 2021 the value of X will be 4 (as 2017 is our base year and its value is 0), the value of Y will be **Y = 1278+ 232.9 (4) = 2209.6**

Odd number of Years

Example 9. The data of sales of Abhilasha Ltd is given for last 8 years. On the basis of the data find trend line using the method of least square and find trend value of 2021.

Year	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**
Sales ‘000’	**80**	**90**	**92**	**83**	**94**	**99**	**92**	**104**

Solution:

Since the number of years is even, so will take the origin as mid point of 2016 and 2017 and further for sake of simplicity we one year is taken as two units (6 Month as 1 unit).

2

Year	X	Sales (Y)	X	XY
2013	-7	80	49	-560
2014	-5	90	25	-450
2015	-3	92	9	-276
2016	-1	83	1	-83

2017	1	94	1	94
2018	3	99	9	297
2019	5	92	25	460
2020	7	104	49	728
	∑X = 0	∑Y =734	$\sum X^2$= 168	∑X Y =210

As ∑X is 0, we can apply short cut method

$$a = \frac{ZY}{n} = \frac{7346}{8} = 91.75$$

$$\frac{ZXY}{\sum K^2} = \frac{210}{168} = 1.25$$

b=

Putting these values in the equation Y = *a* + bX we get
Y = 91.75+ 1.25 X

So if we want to calculate the trend value of the year 2021 the value of X will be 9 (as mid of 2016 and 2017 is our base year and 1 year is taken as 2 units), the value of Y will be **Y = 91.75+ 1.25 (9) = 103**

12.18.3 MERITS OF LEAST SQUARE METHOD

1. There is no subjectiveness in this method as it is based on mathematical calculations.
2. This method is known as method of best fit, reason being the sum of deviations between trend and actual values is zero and sum of square of deviations is least.
3. This method can predict future values, that thing is not possible in moving averate.
4. This method gives us annual growth or decline rate also. The value of ‘b’ in the equation is growth of decline rate. If ‘b’ is positive than it is growth and if it is negative than it is decline. 5. It is based on all the values of the data.

12.18.4 LIMITATIONS OF LEAST SQUARE METHOD:

1. This method involves lot of mathematical calculation, so is difficult for a layman.
2. This method finds trend value only and seasonal, cyclical and irregular variations are completely ignored.
3. If a new value is added to the data, we have to make the complete calculations once again.

12.19 TEST YOUR UNDERSTANDING (D)

1. These are the number of salesmen working in Alpha Ltd:

Year	2011	2012	2013	2014	2015	2016
Salesmen	28	38	46	40	56	60

Fit straight line trend using method of least squares.

2 Fit a straight line trend by Method of least square and estimate the exports of 2021 using the short cut method :

Year	2013	2014	2015	2016	2017	2018	2019	2020
Exports	15	20	24	29	35	45	60	85

3 Determine the equation of straight line which best fits the following data

Year	2012	2013	2014	2015	2016	2017	2018	2019	2020
Value	620	713	833	835	810	745	726	806	861

4 Determine the equation of straight line which best fits the following data

Year	2001	2002	2004	2006	2007
Sales 'Lacs'	5	8	12	20	25

5 Determine the equation using method of least square from number of accidents from the following data and find trend values also.

Year	2001	2002	2003	2004	2005	2006	2007	2008
Accidents	38	40	65	72	69	60	87	95

Answers

1. **Y = 44.67 + 2.97 X**
2. **Y = 39.125 + 4.517 X; value of 2021 – 79.778**
3. **Y = 709.51 + 15.65 X**
4. **Y = 14 + 3.23 X (taking 2004 as year of origin.**
5. **Y = 65.75 + 3.667 X (taking 2004.5 as year of origin).**

Trend Values 40.081, 47.415, 54.749, 62.083, 69.417, 76.751, 84.085, 91.419

12.20 LET US SUM UP

- **Time series analysis is a situation where there are two variables in the problem and out of that one variable is necessarily the time factor.**
- **This analysis is very useful tool for forecasting.**
- **With passage of time there are fluctuations in the items.**
- **These fluctuations are mainly due to four factors called components of time series.**
- **These components are Secular trend, seasonal variations, cyclical variations and irregular variations.**
- **There are two models of time series, these are additive models and multiplicative models.**
- **For finding secular trend we can apply four methods that are Free hand graphic method, Semi Average Method, Moving Average Method and Method of Least Square.**

12.21 KEY TERMS

- **TIME SERIES:**In time series we collect the data related to statistical observations and place such data in chronological order, that means in the order of occurrence of these observations. On the basis of these observations we can try to predict the future values of the observation.
- **SECULAR TREND:** Secular Trends are long-term Trends which normally occurs over it period of 15 to 20 years. Sometime, these trends may show upward results and other time it may show downward results.
- **SEASONAL VARIATIONS:** Seasonal variations are short term variations. These variations occur regularly and their trend is repetitive. These variations may occur on yearly basis, half yearly basis, monthly basis, weekly basis or any other time period basis. These may occur due to climatic condition or due to customs and traditions.
- **CYCLICAL VARIATIONS** . These variations are long Run variations and shows the recurring pattern of rise and decline. These variations are also known as oscillating movements. These variations do not have any fix duration. Sometime one cycle may be complete in 2-3 years, but some other time it may takes 7-8 years to complete for example Trade cycles.
- **IRREGULAR VARIATIONS:** From the name of these variations, it is clear that these variations do not have any definite pattern and are irregular in nature. These variations do

not have any fixed time period and occur due to accidental or random factors like strike, floods, pandemic, war, earthquakes etc.

- **ADDITIVE MODEL OF TIME SERIES:** This model of decomposition assumes that the four elements of time series are not dependent on each other and does not affect each other. Each trend operate independently. So if we have to measure overall trend of the time series, it is combination of all the four elements.
- **MULTIPLICATIVE MODEL:** The Multiplicative model are based on the assumption that all the components of the time series are related to each other and have significant effect on each other. So, if we want to calculate overall trend, it is multiple effect of all the four elements.

12.22 REVIEW QUESTIONS

1. What is time series? Give its significance and limitations.
2. What are components of time series?
3. Give different types of trends in time series.
4. Give multiplicative and additive models of time series.
5. What is free hand curve method?
6. What is semi average method of time series?
7. How predictions are made using method of least square.
8. What is moving average trend. How it is determined.
9. Give various methods of estimating trend along with their respective merits and limitations.

12.22 FURTHER READINGS

1. J. K. Sharma, *Business Statistics,* Pearson Education.
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4. Richard Levin and David S. Rubin, *Statistics for Management,* Prentice Hall of India, New Delhi.
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